1. Evaluate $\int_0^{\frac{\pi}{2}} \sin^5 x\, dx$.

   [Solution]

$$\int_0^{\frac{\pi}{2}} \sin^5 x\, dx = \int_0^{\frac{\pi}{2}} \left(\sin^4 x\right) \sin x\, dx$$
$$= \int_0^{\frac{\pi}{2}} \left(\sin^2 x\right)^2 \sin x\, dx$$
$$= \int_0^{\frac{\pi}{2}} \left(1 - \cos^2 x\right)^2 \sin x\, dx$$

   Let $u = \cos x$, then $du = -\sin x\, dx$.

   When $x = 0$, $u = 1$ and when $x = \frac{\pi}{2}$, $u = 0$.

   Thus $\int_0^{\frac{\pi}{2}} \sin^5 x\, dx = \int_0^{\frac{\pi}{2}} \left(1 - \cos^2 x\right)^2 \sin x\, dx$

$$= -\int_1^0 \left(1 - u^2\right)^2\, du$$
$$= \int_0^1 \left(1 - 2u^2 + u^4\right) du$$
$$= \left(u - \frac{2}{3}u^3 + \frac{1}{5}u^5\right)\Big|_0^1$$
$$= \frac{8}{15}$$

2. Evaluate $\int \frac{\cos^5 x}{\sin x}\, dx$.

   [Solution]

$$\int \frac{\cos^5 x}{\sin x}\, dx = \int \frac{\cos^4 x}{\sin x} \cos x\, dx$$
$$= \int \frac{\left(\cos^2 x\right)^2}{\sin x} \cos x\, dx$$
$$= \int \frac{\left(1 - \sin^2 x\right)^2}{\sin x} \cos x\, dx$$

   Let $u = \sin x$, then $du = \cos x\, dx$.

Thus $\int \frac{\cos^5 x}{\sin x}\,dx = \int \frac{\left(1-\sin^2 x\right)^2}{\sin x}\cos x\,dx$

$$= \int \frac{\left(1-u^2\right)^2}{u}\,du$$

$$= \int \frac{1-2u^2+u^4}{u}\,du$$

$$= \int \left(\frac{1}{u} - 2u + u^3\right) du$$

$$= \ln|u| - u^2 + \frac{1}{4}u^4 + C$$

$$= \ln|\sin x| - \sin^2 x + \frac{1}{4}\sin^4 x + C$$

3. Evaluate $\int_0^{\pi} \cos^4 2x\,dx$.

[Solution]

$$\int_0^{\pi} \cos^4 2x\,dx = \int_0^{\pi} \left(\cos^2 2x\right)^2\,dx$$

$$= \int_0^{\pi} \left(\frac{1+\cos 4x}{2}\right)^2 dx$$

$$= \frac{1}{4}\int_0^{\pi} \left(1 + 2\cos 4x + \cos^2 4x\right) dx$$

$$= \frac{1}{4}\int_0^{\pi} \left(1 + 2\cos 4x + \frac{1+\cos 8x}{2}\right) dx$$

$$= \frac{1}{8}\int_0^{\pi} \left(3 + 4\cos 4x + \cos 8x\right) dx$$

$$= \frac{1}{8}\left(3x + \sin 4x + \frac{1}{8}\sin 8x\right)\Bigg|_0^{\pi}$$

$$= \frac{3}{8}\pi$$

4. Evaluate $\int \sin^3 x \cos^5 x\, dx$.

[Solution] 1

$$\begin{aligned}\int \sin^3 x \cos^5 x\, dx &= \int \left(\sin^3 x \cos^4 x\right)\cos x\, dx \\ &= \int \left[\sin^3 x \left(\cos^2 x\right)^2\right]\cos x\, dx \\ &= \int \left[\sin^3 x \left(1-\sin^2 x\right)^2\right]\cos x\, dx\end{aligned}$$

Let $u = \sin x$, then $du = \cos x\, dx$.

Thus $$\begin{aligned}\int \sin^3 x \cos^5 x\, dx &= \int \left[\sin^3 x \left(1-\sin^2 x\right)^2\right]\cos x\, dx \\ &= \int u^3\left(1-u^2\right)^2 du \\ &= \int u^3\left(1-2u^2+u^4\right) du \\ &= \int \left(u^3-2u^5+u^7\right) du \\ &= \frac{1}{4}u^4-\frac{1}{3}u^6+\frac{1}{8}u^8+C \\ &= \frac{1}{4}\sin^4 x-\frac{1}{3}\sin^6 x+\frac{1}{8}\sin^8 x+C\end{aligned}$$

[Solution] 2

$$\begin{aligned}\int \sin^3 x \cos^5 x\, dx &= \int \left(\sin^2 x \cos^5 x\right)\sin x\, dx \\ &= \int \left[\left(1-\cos^2 x\right)\cos^5 x\right]\sin x\, dx\end{aligned}$$

Let $u = \cos x$, then $du = -\sin x\, dx$.

Thus $$\begin{aligned}\int \sin^3 x \cos^5 x\, dx &= \int \left[\left(1-\cos^2 x\right)\cos^5 x\right]\sin x\, dx \\ &= -\int \left(1-u^2\right)u^5\, du \\ &= -\int \left(u^5-u^7\right) du \\ &= -\frac{1}{6}u^6+\frac{1}{8}u^8+C \\ &= -\frac{1}{6}\cos^6 x+\frac{1}{8}\cos^8 x+C\end{aligned}$$

5. Evaluate $\int \sin^2 x\cos^2 x\, dx$.

[Solution]

$$\begin{aligned}\int \sin^2 x\cos^2 x\, dx &= \int \left(\frac{1-\cos 2x}{2}\right)\left(\frac{1+\cos 2x}{2}\right) dx\\ &= \frac{1}{4}\int \left(1-\cos^2 2x\right) dx\\ &= \frac{1}{4}\int \sin^2 2x\, dx\\ &= \frac{1}{4}\int \frac{1-\cos 4x}{2}\, dx\\ &= \frac{1}{8}\left(x-\frac{1}{4}\sin 4x\right)+C\\ &= \frac{1}{8}x-\frac{1}{32}\sin 4x+C\end{aligned}$$

6. Evaluate $\int \tan^3 x\sec^3 x\, dx$.

[Solution]

$$\begin{aligned}\int \tan^3 x\sec^3 x\, dx &= \int \left(\tan^2 x\sec^2 x\right)\tan x\sec x\, dx\\ &= \int \left[\left(\sec^2 x-1\right)\sec^2 x\right]\tan x\sec x\, dx\end{aligned}$$

Let $u=\sec x$, then $du=\tan x\sec x\, dx$.

Thus $\int \tan^3 x\sec^3 x\, dx = \int \left[\left(\sec^2 x-1\right)\sec^2 x\right]\tan x\sec x\, dx$

$$\begin{aligned}&= \int \left(u^2-1\right)u^2\, du\\ &= \int \left(u^4-u^2\right) du\\ &= \frac{1}{5}u^5-\frac{1}{3}u^3+C\\ &= \frac{1}{5}\sec^5 x-\frac{1}{3}\sec^3 x+C\end{aligned}$$

7. Evaluate $\int \tan^2 x\sec^4 x\, dx$.

[Solution]

$$\int \tan^2 x\sec^4 x\, dx = \int \left(\tan^2 x\sec^2 x\right)\sec^2 x\, dx$$

$$= \int \left[\tan^2 x \left(\tan^2 x + 1 \right) \right] \sec^2 x \, dx$$

Let $u = \tan x$, then $du = \sec^2 x \, dx$.

Thus $\int \tan^2 x \sec^4 x \, dx = \int \left[\tan^2 x \left(\tan^2 x + 1 \right) \right] \sec^2 x \, dx$

$$= \int u^2 \left(u^2 + 1 \right) du$$

$$= \int \left(u^4 + u^2 \right) du$$

$$= \frac{1}{5} u^5 + \frac{1}{3} u^3 + C$$

$$= \frac{1}{5} \tan^5 x + \frac{1}{3} \tan^3 x + C$$

8. Evaluate $\int_0^{\frac{\pi}{4}} \sec^4 x \, dx$.

[Solution]

$$\int_0^{\frac{\pi}{4}} \sec^4 x \, dx = \int_0^{\frac{\pi}{4}} \left(\sec^2 x \right) \sec^2 x \, dx$$

$$= \int_0^{\frac{\pi}{4}} \left(\tan^2 x + 1 \right) \sec^2 x \, dx$$

Let $u = \tan x$, then $du = \sec^2 x \, dx$.

When $x = 0$, $u = 0$ and when $x = \frac{\pi}{4}$, $u = 1$.

Therefore, $\int_0^{\frac{\pi}{4}} \sec^4 x \, dx = \int_0^{\frac{\pi}{4}} \left(\tan^2 x + 1 \right) \sec^2 x \, dx$

$$= \int_0^1 \left(u^2 + 1 \right) du$$

$$= \left(\frac{1}{3} u^3 + u \right) \Bigg|_0^1$$

$$= \frac{4}{3}$$

9. Evaluate $\int \frac{1-\tan^2 x}{1+\tan^2 x}\,dx$.

[Solution]

$\int \frac{1-\tan^2 x}{1+\tan^2 x}\,dx$ (apply the identity tan^2 x + 1 = sec^2 x to the denominator)

(rewrite sec^2 x as cos^2 x)

(apply the identity cos^(2 x) = cos^x – sin^x)

$= \int \cos 2x\,dx$

$= \frac{1}{2}\sin 2x + C$

10. Evaluate $\int \tan^3 x \sec^4 x\,dx$.

[Solution] 1

$$\int \tan^3 x \sec^4 x\,dx = \int \left(\tan^3 x \sec^2 x\right)\sec^2 x\,dx$$
$$= \int \left[\tan^3 x\left(\tan^2 x + 1\right)\right]\sec^2 x\,dx$$

Let $u = \tan x$, then $du = \sec^2 x\,dx$.

Thus
$$\int \tan^3 x \sec^4 x\,dx = \int \left[\tan^3 x\left(\tan^2 x + 1\right)\right]\sec^2 x\,dx$$
$$= \int u^3\left(u^2+1\right)du$$
$$= \int \left(u^5+u^3\right)du$$
$$= \frac{1}{6}u^6 + \frac{1}{4}u^4 + C$$
$$= \frac{1}{6}\tan^6 x + \frac{1}{4}\tan^4 x + C$$

[Solution] 2

$$\int \tan^3 x \sec^4 x\,dx = \int \left(\tan^2 x \sec^3 x\right)\tan x \sec x\,dx$$
$$= \int \left[\left(\sec^2 x - 1\right)\sec^3 x\right]\tan x \sec x\,dx$$

Let $u = \sec x$, then $du = \tan x \sec x\,dx$.

Thus
$$\int \tan^3 x \sec^4 x\,dx = \int \left[\left(\sec^2 x - 1\right)\sec^3 x\right]\tan x \sec x\,dx$$
$$= \int \left(u^2-1\right)u^3\,du$$
$$= \int \left(u^5-u^3\right)du$$
$$= \frac{1}{6}u^6 - \frac{1}{4}u^4 + C$$
$$= \frac{1}{6}\sec^6 x - \frac{1}{4}\sec^4 x + C$$