HW due Week 6 Day 3 (ANSWER KEY)

1. Evaluate $\int \frac{1}{x\sqrt{4-x^2}}\,dx$.

   [Solution]

   Let $x = 2\sin\theta$, where $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$, then $dx = 2\cos\theta\, d\theta$.

   Thus
$$\begin{aligned}\int \frac{1}{x\sqrt{4-x^2}}\,dx &= \int \frac{1}{2\sin\theta\sqrt{4-4\sin^2\theta}}2\cos\theta\, d\theta \\ &= \frac{1}{2}\int \frac{1}{\sin\theta}\,d\theta \\ &= \frac{1}{2}\int \csc\theta\, d\theta \\ &= -\frac{1}{2}\ln\left|\csc\theta + \cot\theta\right| + C \\ &= -\frac{1}{2}\ln\left|\frac{2+\sqrt{4-x^2}}{x}\right| + C\end{aligned}$$

2. Evaluate $\int \frac{1}{\sqrt{x^2+16}}\,dx$.

   [Solution]

   Let $x = 4\tan\theta$, where $-\frac{\pi}{2} < \theta < \frac{\pi}{2}$, then $dx = 4\sec^2\theta\, d\theta$.

   Thus
$$\begin{aligned}\int \frac{1}{\sqrt{x^2+16}}\,dx &= \int \frac{1}{\sqrt{16\tan^2\theta+16}}4\sec^2\theta\, d\theta \\ &= \int \sec\theta\, d\theta \\ &= \ln\left|\sec\theta + \tan\theta\right| + C \\ &= \ln\left|\frac{\sqrt{x^2+16}+x}{4}\right| + C\end{aligned}$$

3. Evaluate $\int_{\sqrt{2}}^{2} \frac{1}{x^3\sqrt{x^2-1}}\,dx$.

   [Solution]

   Let $x = \sec\theta$, where $0 \le \theta < \frac{\pi}{2}$ or $\pi \le \theta < \frac{3\pi}{2}$, then $dx = \sec\theta\tan\theta\, d\theta$.

   When $x = \sqrt{2}$, $\theta = \frac{\pi}{4}$ and when $x = 2$, $\theta = \frac{\pi}{3}$.

Thus $\int_{\sqrt{2}}^{2} \frac{1}{x^3\sqrt{x^2-1}}\,dx = \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{1}{\sec^3\theta\sqrt{\sec^2\theta-1}}\sec\theta\tan\theta\,d\theta$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{1}{\sec^2\theta}\,d\theta$$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \cos^2\theta\,d\theta$$

$$= \int_{\frac{\pi}{4}}^{\frac{\pi}{3}} \frac{1+\cos 2\theta}{2}\,d\theta$$

$$= \left(\frac{1}{2}\theta+\frac{1}{4}\sin 2\theta\right)_{\frac{\pi}{4}}^{\frac{\pi}{3}}$$

$$= \left(\frac{\pi}{6}+\frac{1}{4}\sin\frac{2\pi}{3}\right)-\left(\frac{\pi}{8}+\frac{1}{4}\sin\frac{\pi}{2}\right)$$

$$= \frac{\pi}{24}+\frac{\sqrt{3}}{8}-\frac{1}{4}$$

4. Evaluate $\int_0^{\frac{\pi}{2}} \frac{\cos x}{\sqrt{1+\sin^2 x}}\,dx$.

[Solution]

Let $u=\sin x$, then $du=\cos x\,dx$.

When $x=0$, $u=0$ and when $x=\frac{\pi}{2}$, $u=1$.

Thus $\int_0^{\frac{\pi}{2}} \frac{\cos x}{\sqrt{1+\sin^2 x}}\,dx = \int_0^1 \frac{1}{\sqrt{1+u^2}}\,du$

Let $u=\tan\theta$, where $-\frac{\pi}{2}<\theta<\frac{\pi}{2}$, then $du=\sec^2\theta\,d\theta$.

When $u=0$, $\theta=0$ and when $u=1$, $\theta=\frac{\pi}{4}$.

Thus $\int_0^{\frac{\pi}{2}} \frac{\cos x}{\sqrt{1+\sin^2 x}}\,dx = \int_0^1 \frac{1}{\sqrt{1+u^2}}\,du$

$$= \int_0^{\frac{\pi}{4}} \frac{1}{\sqrt{1+\tan^2\theta}}\sec^2\theta\,d\theta$$

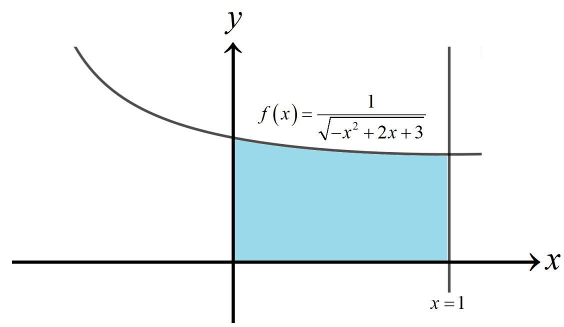
$$= \int_0^{\frac{\pi}{4}} \sec\theta\,d\theta$$

$$= \left(\ln\left|\sec\theta+\tan\theta\right|\right)_0^{\frac{\pi}{4}}$$

$$= \ln\left(\sqrt{2}+1\right)$$

5. Let R be the region bounded by the function $f(x)=\dfrac{1}{\sqrt{-x^2+2x+3}}$ and x-axis on the interval $[0,1]$.

Find the area of the region R.



[Solution] 1

$$A=\int_0^1 \frac{1}{\sqrt{-x^2+2x+3}}\,dx$$

$$\begin{aligned}-x^2+2x+3&=-\left(x^2-2x\right)+3\\&=-\left(x^2-2x+1\right)+1+3\\&=-(x-1)^2+4\end{aligned}$$

Thus $$\begin{aligned}A&=\int_0^1 \frac{1}{\sqrt{-x^2+2x+3}}\,dx\\&=\int_0^1 \frac{1}{\sqrt{4-(x-1)^2}}\,dx\end{aligned}$$

Let $x-1=2\sin\theta$, where $-\frac{\pi}{2}\le\theta\le\frac{\pi}{2}$, then $dx=2\cos\theta\,d\theta$.

When $x=0$, $\theta=-\frac{\pi}{6}$ and when $x=1$, $\theta=0$.

Thus $$\begin{aligned}A&=\int_0^1 \frac{1}{\sqrt{4-(x-1)^2}}\,dx\\&=\int_{-\frac{\pi}{6}}^{0} \frac{1}{\sqrt{4-4\sin^2\theta}}\,2\cos\theta\,d\theta\\&=\int_{-\frac{\pi}{6}}^{0} 1\,d\theta\\&=\frac{\pi}{6}\end{aligned}$$

[Solution] 2

$$\begin{aligned}A&=\int_0^1 \frac{1}{\sqrt{4-(x-1)^2}}\,dx\\&=\int_0^1 \frac{1}{2\sqrt{1-\left(\frac{x-1}{2}\right)^2}}\,dx\end{aligned}$$

Let $u=\frac{x-1}{2}$, then $du=\frac{1}{2}\,dx$.

When $x=0$, $u=-\dfrac{1}{2}$ and when $x=1$, $u=0$.

Thus
$$A=\int_0^1 \frac{1}{2\sqrt{1-\left(\dfrac{x-1}{2}\right)^2}}\,dx$$
$$=\int_{-\frac{1}{2}}^{0}\frac{1}{\sqrt{1-u^2}}\,du$$
$$=\left(\sin^{-1}u\right)_{-\frac{1}{2}}^{0}$$
$$=\frac{\pi}{6}$$

6. Evaluate $\displaystyle\int \frac{1}{\sqrt{1+16x^2}}\,dx$.

[Solution]

Let $x=\dfrac{1}{4}\tan\theta$, where $-\dfrac{\pi}{2}<\theta<\dfrac{\pi}{2}$, then $dx=\dfrac{1}{4}\sec^2\theta\,d\theta$.

Thus
$$\int \frac{1}{\sqrt{1+16x^2}}\,dx=\int\frac{1}{\sqrt{1+\tan^2\theta}}\frac{1}{4}\sec^2\theta\,d\theta$$
$$=\frac{1}{4}\int\sec\theta\,d\theta$$
$$=\frac{1}{4}\ln\left|\sec\theta+\tan\theta\right|+C$$
$$=\frac{1}{4}\ln\left|\sqrt{1+16x^2}+4x\right|+C$$

7. Evaluate $\int_0^{\sqrt{2}} \frac{x^2}{\sqrt{4-x^2}}\,dx$.

[Solution]

Let $x = 2\sin\theta$ where $-\frac{\pi}{2} \le \theta \le \frac{\pi}{2}$, then $dx = 2\cos\theta\,d\theta$.

When $x = 0$, $\theta = 0$ and when $x = \sqrt{2}$, $\theta = \frac{\pi}{4}$.

Thus
$$\int_0^{\sqrt{2}} \frac{x^2}{\sqrt{4-x^2}}\,dx = \int_0^{\frac{\pi}{4}} \frac{4\sin^2\theta}{\sqrt{4-4\sin^2\theta}}2\cos\theta\,d\theta$$
$$= \int_0^{\frac{\pi}{4}} 4\sin^2\theta\,d\theta$$
$$= 4\int_0^{\frac{\pi}{4}} \frac{1-\cos 2\theta}{2}\,d\theta$$
$$= \left(2\theta - \sin 2\theta\right)_0^{\frac{\pi}{4}}$$
$$= \frac{\pi}{2} - 1$$

8. Evaluate $\int \frac{1}{x^2\sqrt{9x^2-1}}\,dx$.

[Solution]

Let $x = \frac{1}{3}\sec\theta$ where $0 \le \theta < \frac{\pi}{2}$ or $\pi \le \theta < \frac{3\pi}{2}$, then $dx = \frac{1}{3}\sec\theta\tan\theta\,d\theta$.

Thus
$$\int \frac{1}{x^2\sqrt{9x^2-1}}\,dx = \int \frac{1}{\frac{1}{9}\sec^2\theta\sqrt{\sec^2\theta-1}}\frac{1}{3}\sec\theta\tan\theta\,d\theta$$
$$= 3\int \frac{1}{\sec\theta}\,d\theta$$
$$= 3\int \cos\theta\,d\theta$$
$$= 3\sin\theta + C$$
$$= \frac{\sqrt{9x^2-1}}{x} + C$$

9. Evaluate $\int \sqrt{5+4x-x^2}\,dx$.

[Solution]

$$\begin{aligned}5+4x-x^2 &= -\left(x^2-4x\right)+5\\ &= -\left(x^2-4x+4\right)+4+5\\ &= -\left(x-2\right)^2+9\end{aligned}$$

Let $x-2=3\sin\theta$, where $-\frac{\pi}{2}\le\theta\le\frac{\pi}{2}$, then $dx=3\cos\theta\,d\theta$.

Thus
$$\begin{aligned}\int\sqrt{5+4x-x^2}\,dx &= \int\sqrt{9-\left(x-2\right)^2}\,dx\\ &= \int 3\cos\theta\sqrt{9-9\sin^2\theta}\,d\theta\\ &= 9\int\cos^2\theta\,d\theta\\ &= 9\int\frac{1+\cos 2\theta}{2}\,d\theta\\ &= \frac{9}{2}\theta+\frac{9}{4}\sin 2\theta+C\\ &= \frac{9}{2}\theta+\frac{9}{2}\sin\theta\cos\theta+C\\ &= \frac{9}{2}\sin^{-1}\left(\frac{x-2}{3}\right)+\frac{9}{2}\cdot\frac{x-2}{3}\cdot\frac{\sqrt{9-\left(x-2\right)^2}}{3}+C\\ &= \frac{9}{2}\sin^{-1}\left(\frac{x-2}{3}\right)+\frac{x-2}{2}\sqrt{9-\left(x-2\right)^2}+C\end{aligned}$$