ANSWER KEY for homework due Week 6 Mon, Feb 26

Name : ______________________________

1. Evaluate $\int \ln\left(x+\sqrt{1+x^2}\right)\,dx$.

[Solution]

Let $u=\ln\left(x+\sqrt{1+x^2}\right)$ and $dv=dx$,

then
$$du=\frac{1}{x+\sqrt{1+x^2}}\cdot\left(1+\frac{1}{2\sqrt{1+x^2}}\cdot 2x\right)\,dx$$
$$=\frac{1}{x+\sqrt{1+x^2}}\cdot\frac{\sqrt{1+x^2}+x}{\sqrt{1+x^2}}\,dx$$
$$=\frac{1}{\sqrt{1+x^2}}\,dx$$

and $v=x$.

Thus $\int \ln\left(x+\sqrt{1+x^2}\right)\,dx=x\ln\left(x+\sqrt{1+x^2}\right)-\int\frac{x}{\sqrt{1+x^2}}\,dx$

For $\int\frac{x}{\sqrt{1+x^2}}\,dx$, let $w=1+x^2$, then $dw=2x\,dx$.

Thus
$$\int\frac{x}{\sqrt{1+x^2}}\,dx=\int\frac{1}{2\sqrt{w}}\,dw$$
$$=\sqrt{w}+C$$
$$=\sqrt{1+x^2}+C$$

Therefore, $\int \ln\left(x+\sqrt{1+x^2}\right)\,dx=x\ln\left(x+\sqrt{1+x^2}\right)-\sqrt{1+x^2}+C$

2. Evaluate $\int x\tan^2 x\,dx$.

[Solution]

$$\int x\tan^2 x\,dx=\int x\left(\sec^2 x-1\right)\,dx$$
$$=\int x\sec^2 x\,dx-\int x\,dx$$

For $\int x\sec^2 x\,dx$, let $u=x$ and $dv=\sec^2 x\,dx$,

then $du=dx$ and $v=\tan x$.

Thus
$$\int x\sec^2 x\,dx=x\tan x-\int\tan x\,dx$$
$$=x\tan x-\int\frac{\sin x}{\cos x}\,dx$$

For $\int \frac{\sin x}{\cos x}\,dx$, let $w = \cos x$, then $dw = -\sin x\,dx$.

Thus $\int \tan x\,dx = \int \frac{\sin x}{\cos x}\,dx$

$$= -\int \frac{1}{w}\,dw$$
$$= -\ln|w| + C$$
$$= -\ln|\cos x| + C$$

Therefore, $\int x\tan^2 x\,dx = \int x\sec^2 x\,dx - \int x\,dx$

$$= \left(x\tan x - \int \tan x\,dx\right) - \frac{1}{2}x^2$$
$$= x\tan x + \ln|\cos x| - \frac{1}{2}x^2 + C$$

3. Evaluate $\int \cos\sqrt{x}\,dx$

[Solution]

Let $w = \sqrt{x}$, then $dw = \frac{1}{2\sqrt{x}}\,dx$.

Thus $\int \cos\sqrt{x}\,dx = 2\int w\cos w\,dw$

For $\int w\cos w\,dw$, let $u = w$ and $dv = \cos w\,dw$,

then $du = dw$ and $v = \sin w$.

Thus $\int w\cos w\,dw = w\sin w - \int \sin w\,dw$

$$= w\sin w + \cos w + c$$

Therefore, $\int \cos\sqrt{x}\,dx =$ 2 [sqrt(x) sin (sqrt(x) + cos (sqrt(x))] + C

4. Evaluate $\int x^2 (\ln x)^2 \, dx$.

[Solution]

Let $u = (\ln x)^2$ and $dv = x^2 \, dx$,

then $du = \frac{2\ln x}{x} \, dx$ and $v = \frac{1}{3}x^3$.

Thus $\int x^2 (\ln x)^2 \, dx = \frac{1}{3}x^3 (\ln x)^2 - \frac{2}{3}\int x^2 \ln x \, dx$

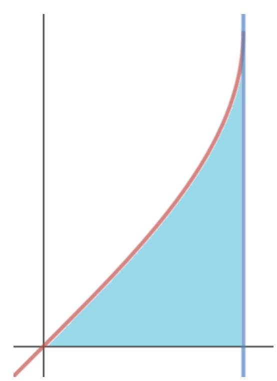
For $\int x^2 \ln x \, dx$, let $s = \ln x$ and $dt = x^2 \, dx$,

then $ds = \frac{1}{x} \, dx$ and $t = \frac{1}{3}x^3$.

Thus $\int x^2 \ln x \, dx = \frac{1}{3}x^3 \ln x - \frac{1}{3}\int x^2 \, dx$

Therefore,
$$\begin{aligned}\int x^2 (\ln x)^2 \, dx &= \frac{1}{3}x^3 (\ln x)^2 - \frac{2}{3}\int x^2 \ln x \, dx \\ &= \frac{1}{3}x^3 (\ln x)^2 - \frac{2}{3}\left(\frac{1}{3}x^3 \ln x - \frac{1}{3}\int x^2 \, dx\right) \\ &= \frac{1}{3}x^3 (\ln x)^2 - \frac{2}{9}x^3 \ln x + \frac{2}{27}x^3 + C\end{aligned}$$

5. Consider the graph of the function $f(x) = \sin^{-1} x$. Let R be the region bounded by $y = f(x)$ and x-axis on the interval $[0,1]$.



Evaluate the **area** of R.

[Solution]

$A = \int_0^1 \sin^{-1} x \, dx$

Let $u = \sin^{-1} x$ and $dv = dx$,

then $du = \frac{1}{\sqrt{1-x^2}} \, dx$ and $v = x$.

Thus $\int \sin^{-1} x \, dx = x \sin^{-1} x - \int \frac{x}{\sqrt{1-x^2}} \, dx$

For $\int \frac{x}{\sqrt{1-x^2}}\,dx$, let $w = 1-x^2$, then $dw = -2x\,dx$.

Thus $\int \frac{x}{\sqrt{1-x^2}}\,dx = -\int \frac{1}{2\sqrt{w}}\,dw$

$$= -\sqrt{w} + C$$

$$= -\sqrt{1-x^2} + c$$

Therefore, $\int \sin^{-1} x\,dx = x\sin^{-1} x - \int \frac{x}{\sqrt{1-x^2}}\,dx$

$$= x\sin^{-1} x + \sqrt{1-x^2} + C$$

As a result, $A = \int_0^1 \sin^{-1} x\,dx$

$$= \left(x\sin^{-1} x + \sqrt{1-x^2}\right)\Big|_0^1$$

$$= \sin^{-1} 1 - 1$$

$$= \frac{\pi}{2} - 1$$

6. Evaluate $\int \cos(\ln x)\,dx$.

[Solution]

Let $u = \cos(\ln x)$ and $dv = dx$,

then $du = \frac{-\sin(\ln x)}{x}\,dx$ and $v = x$.

Thus $\int \cos(\ln x)\,dx = x\cos(\ln x) + \int \sin(\ln x)\,dx$

For $\int \sin(\ln x)\,dx$, let $s = \sin(\ln x)$ and $dt = dx$,

then $ds = \frac{\cos(\ln x)}{x}\,dx$ and $t = x$.

Thus $\int \sin(\ln x)\,dx = x\sin(\ln x) - \int \cos(\ln x)\,dx$

Therefore, $\int \cos(\ln x)\,dx = x\cos(\ln x) + \int \sin(\ln x)\,dx$

$$= x\cos(\ln x) + \left[x\sin(\ln x) - \int \cos(\ln x)\,dx\right]$$

As a result, $\int \cos(\ln x)\,dx = \frac{x\cos(\ln x) + x\sin(\ln x)}{2} + C$

7. There are parts (a)-(f).

a) To derive the formula for **Integration by Parts** we used which of the following theorems?

1) The Fundamental Theorem of Calculus.

2) The Product Rule of Differentiation.

3) The Chain Rule of Differentiation.

4) The Mean Value Theorem

b) Evaluate $\int_0^{\frac{\pi}{2}} x\cos 2x\,dx$.

[Solution]

Let $u = x$ and $dv = \cos 2x\,dx$,

then $du = dx$ and $v = \frac{1}{2}\sin 2x$.

Thus
$$\begin{aligned}\int_0^{\frac{\pi}{2}} x\cos 2x\,dx &= \frac{1}{2}\left(x\sin 2x\right)_0^{\frac{\pi}{2}} - \frac{1}{2}\int_0^{\frac{\pi}{2}} \sin 2x\,dx \\ &= \frac{1}{2}\left[\frac{\pi}{2}\sin\pi - 0\right] - \frac{1}{2}\left(-\frac{1}{2}\cos 2x\right)_0^{\frac{\pi}{2}} \\ &= \frac{1}{4}\left(\cos\pi - \cos 0\right) \\ &= -\frac{1}{2}\end{aligned}$$

c) Suppose that $f(1)=2$, $f(4)=7$, $f'(1)=5$, $f'(4)=3$ and f'' is continuous. Evaluate $\int_1^4 x f''(x)\,dx$.

[Solution]

Let $u=x$ and $dv=f''(x)\,dx$,

then $du=dx$ and $v=f'(x)$.

Thus $\int_1^4 x f''(x)\,dx = xf'(x)\Big|_1^4 - \int_1^4 f'(x)\,dx$

$$=\left[4f'(4)-f'(1)\right]-\left[f(4)-f(1)\right]$$

$$=2$$

d) Evaluate $\int \tan^{-1} x\,dx$.

[Solution]

Let $u=\tan^{-1} x$ and $dv=dx$,

then $du=\dfrac{1}{1+x^2}\,dx$ and $v=x$.

Thus $\int \tan^{-1} x\,dx = x\tan^{-1} x - \int \dfrac{x}{1+x^2}\,dx$.

For $\int \dfrac{x}{1+x^2}\,dx$, let $w=1+x^2$, then $dw=2x\,dx$.

Therefore $\int \tan^{-1} x\,dx = x\tan^{-1} x - \int \dfrac{x}{1+x^2}\,dx$

$$= x\tan^{-1} x - \frac{1}{2}\int \frac{1}{w}\,dw$$

$$= x\tan^{-1} x - \frac{1}{2}\ln|w| + C$$

$$= x\tan^{-1} x - \frac{1}{2}\ln\left(1+x^2\right) + C$$

Note that $\ln\left|1+x^2\right| = \ln\left(1+x^2\right)$ here because $1+x^2>0$ for all real numbers x.

e) Evaluate $\int e^x \cos x \, dx$.

[Solution]

Let $u = e^x$ and $dv = \cos x \, dx$

Then $du = e^x \, dx$ and $v = \sin x$

Thus $\int e^x \cos x \, dx = e^x \sin x - \int e^x \sin x \, dx$

For $\int e^x \sin x \, dx$

Let $s = e^x$ and $dt = \sin x \, dx$

Then $ds = e^x \, dx$ and $t = -\cos x$

Thus $\int e^x \sin x \, dx = -e^x \cos x + \int e^x \cos x \, dx$

As a result,
$$\begin{aligned}\int e^x \cos x \, dx &= e^x \sin x - \int e^x \sin x \, dx \\ &= e^x \sin x - \left[-e^x \cos x + \int e^x \cos x \, dx \right] \\ &= e^x \sin x + e^x \cos x - \int e^x \cos x \, dx\end{aligned}$$

Therefore, $\int e^x \cos x \, dx = \frac{e^x}{2}(\sin x + \cos x) + C$

[Alternative Solution]

Let $u = \cos x$ and $dv = e^x \, dx$

Then $du = -\sin x \, dx$ and $v = e^x$

Thus $\int e^x \cos x \, dx = e^x \cos x + \int e^x \sin x \, dx$

For $\int e^x \sin x \, dx$

Let $s = \sin x$ and $dt = e^x \, dx$

Then $ds = \cos x \, dx$ and $t = e^x$

Thus $\int e^x \sin x \, dx = e^x \sin x - \int e^x \cos x \, dx$

As a result,
$$\begin{aligned}\int e^x \cos x \, dx &= e^x \cos x + \int e^x \sin x \, dx \\ &= e^x \cos x + \left[e^x \sin x - \int e^x \cos x \, dx \right] \\ &= e^x \cos x + e^x \sin x - \int e^x \cos x \, dx\end{aligned}$$

Therefore, $\int e^x \cos x \, dx = \frac{e^x}{2}(\sin x + \cos x) + C$

f) A particle that moves along a straight line has velocity $v(t) = t^3 e^{-t}$ meters per second after t seconds. How far will it travel during the first t seconds?

[Solution]

The particle will move $s(t) = \int_0^t v(x)\, dx$ meters for the first t seconds.

For $\int x^3 e^{-x}\, dx$

Let $u = x^3$ and $dv = e^{-x}\, dx$

Then $du = 3x^2\, dx$ and $v = -e^{-x}$

Thus $\int x^3 e^{-x}\, dx = -x^3 e^{-x} + 3\int x^2 e^{-x}\, dx$

For $\int x^2 e^{-x}\, dx$

Let $m = x^2$ and $dn = e^{-x}\, dx$

Then $dm = 2x\, dx$ and $n = -e^{-x}$

Thus $\int x^2 e^{-x}\, dx = -x^2 e^{-x} + 2\int x e^{-x}\, dx$

Therefore,
$$\begin{aligned}\int x^3 e^{-x}\, dx &= -x^3 e^{-x} + 3\int x^2 e^{-x}\, dx \\ &= -x^3 e^{-x} + 3\left[-x^2 e^{-x} + 2\int x e^{-x}\, dx\right] \\ &= -x^3 e^{-x} - 3x^2 e^{-x} + 6\int x e^{-x}\, dx\end{aligned}$$

For $\int x e^{-x}\, dx$

Let $f = x$ and $dg = e^{-x}\, dx$

Then $df = dx$ and $g = -e^{-x}$

Thus $\int x e^{-x}\, dx = -x e^{-x} + \int e^{-x}\, dx$

Therefore,
$$\begin{aligned}\int x^3 e^{-x}\, dx &= -x^3 e^{-x} - 3x^2 e^{-x} + 6\int x e^{-x}\, dx \\ &= -x^3 e^{-x} - 3x^2 e^{-x} + 6\left[-x e^{-x} + \int e^{-x}\, dx\right] \\ &= -x^3 e^{-x} - 3x^2 e^{-x} - 6x e^{-x} + 6\int e^{-x}\, dx \\ &= -x^3 e^{-x} - 3x^2 e^{-x} - 6x e^{-x} - 6e^{-x} + C\end{aligned}$$

As a result,
$$\begin{aligned}s(t) &= \int_0^t v(x)\, dx \\ &= \left(-t^3 e^{-t} - 3t^2 e^{-t} - 6t e^{-t} - 6e^{-t} + C\right) - \left(-6 + C\right) \\ &= -t^3 e^{-t} - 3t^2 e^{-t} - 6t e^{-t} - 6e^{-t} + 6\end{aligned}$$