Name : ________________________________

Recall

Reverse **Chain Rule** to get **Substitution Rule**.

$$\frac{d}{dx}\left[f\left(g\left(x\right)\right)\right]=f'\left(g\left(x\right)\right)g'\left(x\right)$$

$$\int f'\left(g\left(x\right)\right)g'\left(x\right)\,dx=f\left(g\left(x\right)\right)+C$$

Let $u=g\left(x\right)$, then $du=g'\left(x\right)dx$. Thus

$$\int f'\left(g\left(x\right)\right)g'\left(x\right)\,dx=\int f'\left(u\right)\,du=f\left(u\right)+C=f\left(g\left(x\right)\right)+C.$$

Integration by Parts

Reverse **Product Rule** to get **Integration by Parts**.

$$\frac{d}{dx}\left[u\left(x\right)v\left(x\right)\right]=u'\left(x\right)v\left(x\right)+u\left(x\right)v'\left(x\right)$$

$$\int u'\left(x\right)v\left(x\right)\,dx+\int u\left(x\right)v'\left(x\right)\,dx=u\left(x\right)v\left(x\right)+C$$

$$\int u\left(x\right)v'\left(x\right)\,dx=u\left(x\right)v\left(x\right)-\int u'\left(x\right)v\left(x\right)\,dx$$

Since $\frac{dv}{dx}=v'\left(x\right)$ and $\frac{du}{dx}=u'\left(x\right)$, we can obtain

$$\int u\,dv=uv-\int v\,du.$$

Integration by Parts

Suppose that u and v are differentiable functions. Then,

$$\int u\,dv=uv-\int v\,du.$$

Integration by Parts is an integration technique for evaluating integrals of **product of functions**.

Integration by Parts

To use Integration by Parts, one should

- Choose u and dv. Note: dv should be **easy to integrate**.
- Evaluate du and v.
- Apply the formula.

Integration by Parts for Definite Integrals

Let u and v be differentiable. Then,

$$\int_a^b u\,dv = uv\Big|_a^b - \int_a^b v\,du.$$

TASK 1: First attempt on your own. Then follow pg 473, Sec 7.1 Ex 2 to evaluate the indefinite integral. Compute the definite integral on your own. Check your answer with WolframAlpha.

Evaluate $\int_1^e \ln x\,dx$.

Recall

Integration by Parts

Suppose that u and v are differentiable functions. Then,

$$\int u\, dv = uv - \int v\, du.$$

Repeated Use of Integration by Parts

[Type 1] Use Integration by Parts AGAIN.

TASK 2: First attempt on your own. This requires multiple applications of integration by parts. Then follow the solution given on pg 474 Sec 7.1 Example 3.
Evaluate $\int x^2 e^x\, dx$.

[Type 2] Use Integration by Parts AGAIN + MERGE.

TASK 3: First attempt on your own (it does take multiple steps using Calc II methods).
Then follow the solution given on pg 474 Sec 7.1 Ex 4.
Evaluate $\int e^x \sin x \, dx$.

Products of tangent (even power) and secant (odd power) - u-sub doesn't work.

TASK 4: Evaluate the antiderivative of (sec^3 x) on your own and by copying pg 483 Sec 7.2 Ex 8. Hint: You've already evaluated the antiderivative for (sec x) in your last reading homework: https://egunawan.github.io/spring18/notes/notes7_2part2.pdf
or you can look this up at the top of page 483 (Don't memorize this antiderivative! This antiderivative will either be given to you or you'll be asked to evaluate it on tests).

Complete problems 1-6. Show all your work. You may use hints and any technology/sources/people.

1. Evaluate $\int \ln\left(x+\sqrt{1+x^2}\right)\,dx$.

[Solution]

Hint: Integration by parts. You have no choice but to let $u=\ln\left(x+\sqrt{1+x^2}\right)$ and $dv=dx$.

$$\int \ln\left(x+\sqrt{1+x^2}\right)\,dx = x\ln\left(x+\sqrt{1+x^2}\right)-\int \frac{x}{\sqrt{1+x^2}}\,dx$$

To solve the right-most integral, do substitution $w=1+x^2$.

Get $\int \ln\left(x+\sqrt{1+x^2}\right)\,dx = x\ln\left(x+\sqrt{1+x^2}\right)-\sqrt{1+x^2}+C$

2. Evaluate $\int x\tan^2 x\,dx$.

[Solution]

Use trig identity to get $\int x\tan^2 x\,dx = \int x\left(\sec^2 x - 1\right)dx$

$$= \int x\sec^2 x\,dx - \int x\,dx$$

Use integration by parts to get $\int x\sec^2 x\,dx = x\tan x - \int \tan x\,dx$

$$= x\tan x - \int \frac{\sin x}{\cos x}\,dx$$

Evaluate $\int \frac{\sin x}{\cos x}\,dx$ by substitution (let $w = \cos x$, then $dw = -\sin x\,dx$.)

$$\int x\tan^2 x\,dx = \int x\sec^2 x\,dx - \int x\,dx$$

$$= x\tan x + \ln\left|\cos x\right| - \frac{1}{2}x^2 + C$$

3. Evaluate $\int \cos\sqrt{x}\ dx$

[Solution]

Substitute $w=\sqrt{x}$ and $dw=\dfrac{1}{2\sqrt{x}}\ dx$ to get $\int \cos\sqrt{x}\ dx = 2\int w\cos w\ dw$

Use integration by parts to get $\int w\cos w\ dw = w\sin w - \int \sin w\ dw$

$= w\sin w + \cos w + c$

$\int \cos\sqrt{x}\ dx$= 2 [sqrt(x) sin (sqrt(x) + cos (sqrt(x))] + C

4. Evaluate $\int x^2 (\ln x)^2 \, dx$.

[Solution]

Use integration by parts to get $\int x^2 (\ln x)^2 \, dx = \frac{1}{3}x^3 (\ln x)^2 - \frac{2}{3}\int x^2 \ln x \, dx$

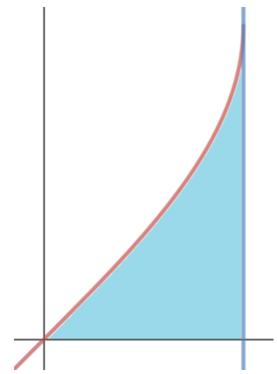
Use integration by parts to get $\int x^2 \ln x \, dx = \frac{1}{3}x^3 \ln x - \frac{1}{3}\int x^2 \, dx$

Therefore, $\int x^2 (\ln x)^2 \, dx = \frac{1}{3}x^3 (\ln x)^2 - \frac{2}{3}\int x^2 \ln x \, dx$

$$= \frac{1}{3}x^3(\ln x)^2 - \frac{2}{9}x^3 \ln x + \frac{2}{27}x^3 + C$$

5. Consider the graph of the function $f(x) = \sin^{-1} x$. Let R be the region bounded by $y = f(x)$ and x-axis on the interval $[0,1]$.

   Evaluate the **area** of R.



[Solution]

Let $u = \sin^{-1} x$ and $dv = dx$,

then $du = \frac{1}{\sqrt{1-x^2}}\,dx$ (verify this is by doing integration by inverse trig substitution).

Therefore, $\int \sin^{-1} x\,dx = x\sin^{-1} x - \int \frac{x}{\sqrt{1-x^2}}\,dx$ by above explanation

$= x\sin^{-1} x + \sqrt{1-x^2} + C$ by using substitution $w = 1 - x^2$

The area is $A = \int_0^1 \sin^{-1} x\,dx$

$$= \frac{\pi}{2} - 1$$

6. Evaluate $\int \cos(\ln x)\, dx$. (Hint: the same strategy as Sec 7.1 Ex 4 on pg 474-475.)

[Solution]

Use integration by parts (no choice but to let $u = \cos(\ln x)$ and $dv = dx$),

and get $\int \cos(\ln x)\, dx = x\cos(\ln x) + \int \sin(\ln x)\, dx$

Use integration by parts again to evaluate $\int \sin(\ln x)\, dx$

Combine the two $\int \cos(\ln x)\, dx$. See Sec 7.1 Example 4 on page 474-475.

Get $\int \cos(\ln x)\, dx = \dfrac{x\cos(\ln x) + x\sin(\ln x)}{2} + C$

7. Complete part (a) and at least **ONE** of parts (b)-(f).

a) To derive the formula for **Integration by Parts** we used which of the following theorems?

i. The Fundamental Theorem of Calculus.

ii. The Product Rule of Differentiation.

iii. The Chain Rule of Differentiation.

iv. The Mean Value Theorem

b) Evaluate $\int_0^{\frac{\pi}{2}} x\cos 2x\, dx$. Hint: integration by parts.

c) Suppose that $f(1)=2$, $f(4)=7$, $f'(1)=5$, $f'(4)=3$ and f'' is continuous. Evaluate $\int_1^4 x f''(x)\, dx$.

d) Evaluate $\int \tan^{-1} x\, dx$.

e) Evaluate $\int e^x \cos x\, dx$.

f) A particle that moves along a straight line has velocity $v(t) = t^3 e^{-t}$ meters per second after t seconds. How far will it travel during the first t seconds?