Section 7.8_1 **Learning Activity (KEY)** MATH 1152Q

Name : ______________________________

1. Evaluate $\int_0^\infty e^{-2x}\ dx$.

   [Solution]

   $$\begin{aligned}\int_0^\infty e^{-2x}\ dx &= \lim_{t\to\infty}\int_0^t e^{-2x}\ dx \\ &= \lim_{t\to\infty}\left(\frac{1}{-2}e^{-2x}\right)_0^t \\ &= \frac{1}{2}\lim_{t\to\infty}\left(-e^{-2t}+1\right)_0^t \\ &= \frac{1}{2}\lim_{t\to\infty}\left(-\frac{1}{e^{2t}}+1\right)=\frac{1}{2}\end{aligned}$$

2. Evaluate $\int_1^\infty \frac{1}{\sqrt{x}}\ dx$.

   [Solution]

   $$\begin{aligned}\int_1^\infty \frac{1}{\sqrt{x}}\ dx &= \lim_{t\to\infty}\int_1^t \frac{1}{\sqrt{x}}\ dx \\ &= \lim_{t\to\infty}\left(2\sqrt{x}\right)_1^t \\ &= 2\lim_{t\to\infty}\left(\sqrt{t}-1\right) \\ &= \infty\end{aligned}$$

   The improper integral $\int_1^\infty \frac{1}{\sqrt{x}}\ dx$ diverges.

3. Evaluate $\int_0^\infty \sin^2 x\ dx$.

   [Solution]

   $$\begin{aligned}\int_0^\infty \sin^2 x\ dx &= \lim_{t\to\infty}\int_0^t \sin^2 x\ dx \\ &= \lim_{t\to\infty}\int_0^t\left(\frac{1-\cos 2x}{2}\right)dx \\ &= \frac{1}{2}\lim_{t\to\infty}\int_0^t(1-\cos 2x)\ dx \\ &= \frac{1}{2}\lim_{t\to\infty}\left(x-\frac{1}{2}\sin 2x\right)_0^t \\ &= \frac{1}{2}\lim_{t\to\infty}\left(t-\frac{1}{2}\sin 2t\right) \text{ does not exist}\end{aligned}$$

The improper integral $\int_0^\infty \sin^2 x\ dx$ diverges.

4. Evaluate $\int_2^{\infty} \frac{1}{x^2+2x-3}\,dx$.

[Solution] Option 1 using trig substitution

Note that $\frac{1}{x^2+2x-3} = \frac{1}{(x+1)^2-4}$

Let $x+1 = 2\sec\theta$, where $0 \le \theta < \frac{\pi}{2}$, then $dx = 2\sec\theta\tan\theta\,d\theta$.

When $x=2$, $\theta = \sec^{-1}\left(\frac{3}{2}\right)$ and when $x=t$, $\theta = \sec^{-1}\left(\frac{t+1}{2}\right)$.

Thus
$$\begin{aligned}
\int_2^{\infty} \frac{1}{x^2+2x-3}\,dx &= \lim_{t\to\infty}\int_2^{t} \frac{1}{x^2+2x-3}\,dx \\
&= \lim_{t\to\infty}\int_2^{t} \frac{1}{(x+1)^2-4}\,dx \\
&= \lim_{t\to\infty}\int_{\sec^{-1}\left(\frac{3}{2}\right)}^{\sec^{-1}\left(\frac{t+1}{2}\right)} \frac{1}{4\tan^2\theta}\cdot 2\sec\theta\tan\theta\,d\theta \\
&= \frac{1}{2}\lim_{t\to\infty}\int_{\sec^{-1}\left(\frac{3}{2}\right)}^{\sec^{-1}\left(\frac{t+1}{2}\right)} \csc\theta\,d\theta \\
&= -\frac{1}{2}\lim_{t\to\infty}\left(\ln\left|\csc\theta+\cot\theta\right|\right)_{\sec^{-1}\left(\frac{3}{2}\right)}^{\sec^{-1}\left(\frac{t+1}{2}\right)} \\
&= -\frac{1}{2}\lim_{t\to\infty}\left(\ln\left|\frac{t+3}{\sqrt{(t+1)^2-4}}\right| - \ln\sqrt{5}\right) = \frac{1}{4}\ln 5
\end{aligned}$$

[Solution] option 2 using partial fraction decomposition

Note that $\frac{1}{x^2+2x-3} = \frac{1}{(x+3)(x-1)} = -\frac{1}{4}\cdot\frac{1}{x+3} + \frac{1}{4}\cdot\frac{1}{x-1}$

$$\begin{aligned}
\int_2^{\infty} \frac{1}{x^2+2x-3}\,dx &= \lim_{t\to\infty}\int_2^{t} \frac{1}{x^2+2x-3}\,dx \\
&= \lim_{t\to\infty}\int_2^{t} \left(-\frac{1}{4}\cdot\frac{1}{x+3} + \frac{1}{4}\cdot\frac{1}{x-1}\right) dx \\
&= \frac{1}{4}\lim_{t\to\infty}\left[-\int_2^{t} \frac{1}{x+3}\,dx + \int_2^{t}\frac{1}{x-1}\,dx\right] \\
&= \frac{1}{4}\lim_{t\to\infty}\left[-\left(\ln|x+3|\right)_2^t + \left(\ln|x-1|\right)_2^t\right] \\
&= \frac{1}{4}\lim_{t\to\infty}\left[-\ln|t+3| + \ln 5 + \ln|t-1|\right] \\
&= \frac{1}{4}\lim_{t\to\infty}\left[\ln\left|\frac{t-1}{t+3}\right| + \ln 5\right] = \frac{1}{4}\ln 5
\end{aligned}$$