10.2 Calculus with Parametric Curves

Derivative of Parametric Curves.

$$\frac{dy}{dx} = \frac{\frac{dy}{dt}}{\frac{dx}{dt}} \qquad \text{if } \frac{dx}{dt} \neq 0.$$

Arc Length. If a curve C is described by the parametric equation $x = f(t)$, $y = g(t)$ for $\alpha \leq t \leq \beta$, where f' and g' are continuous on $[\alpha, \beta]$ and C is traversed exactly once as t increases from α to β, then the length of C is

$$L = \int_{\alpha}^{\beta} \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

Surface Area. If the curve given by the parametric equations $x = f(t)$, $y = g(t)$, $\alpha \leq t \leq \beta$ is rotated about the x-axis, where f' and g' are continuous and $g(t) \geq 0$, then the area of the resulting surface is given by

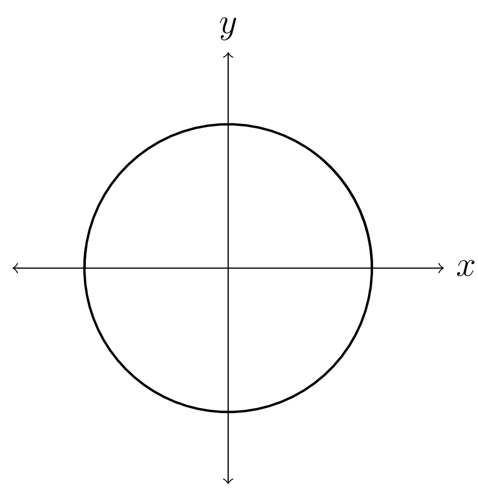
$$S = \int_{\alpha}^{\beta} 2\pi y \sqrt{\left(\frac{dx}{dt}\right)^2 + \left(\frac{dy}{dt}\right)^2}\, dt$$

1. **Example:** On the parametric curve $(x, y) = (-\sin(2t), \cos(2t))$, find the points on the curve with a vertical tangent or a horizontal tangent.

   Thinking about the problem:

   How should I determine tangents of points on the curve? Have I seen a problem similar to this one before? If so, how did I approach it?

   I know that $\sin^2(2t) + \cos^2(2t) = 1$, so $x^2 + y^2 = 1$ and I know that my curve should look like a circle with radius 1 centered at the origin.

   How do I find the Horizontal (HT) and Vertical (VT) tangent lines to the curve? By definition, HT occur when $\frac{dy}{dt} = 0$ (provided that $\frac{dx}{dt} \neq 0$) and VT occur when $\frac{dx}{dt} = 0$ (provided that $\frac{dy}{dt} \neq 0$).

   Since the curve I am considering is a circle, I should expect the points with a vertical or horizontal tangent line should be $(1, 0)$, $(0, 1)$, $(-1, 0)$, and $(0, -1)$.

Doing the problem:

Horizontal tangent lines occur when $\frac{dy}{dt} = 0$ provided that $\frac{dx}{dt} \neq 0$. So I need only find when $\frac{dy}{dt} = y'(t) = 0$. I find

$$\begin{aligned}
y'(t) &= 0 \\
-2\sin(2t) &= 0 \\
\sin(2t) &= 0 \\
2t &= \sin^{-1}(0) \\
2t &= n\pi \qquad \text{for } n = 0, \pm 1, \pm 2, \ldots \\
t &= n\frac{\pi}{2} \qquad \text{for } n = 0, \pm 1, \pm 2, \ldots.
\end{aligned}$$

So the horizontal tangent lines occur at the points:

$$\begin{aligned}
\left(x\left(n\frac{\pi}{2}\right), y\left(n\frac{\pi}{2}\right)\right) &= \left(-\sin\left(2 \cdot n\frac{\pi}{2}\right), \cos\left(2 \cdot n\frac{\pi}{2}\right)\right) \\
&= \left(-\sin(n\pi), \cos(n\pi)\right) \\
&= (0, 1) \text{ and } (0, -1).
\end{aligned}$$

Vertical tangent lines occur when $\frac{dx}{dt} = 0$ and $\frac{dy}{dt} \neq 0$. So to find the vertical tangents, I find

$$\begin{aligned}
\frac{dx}{dt} &= x'(t) = 0 \\
-2\cos(2t) &= 0 \\
\cos(2t) &= 0 \\
2t &= \cos^{-1}(0) \\
2t &= n\frac{\pi}{2} \qquad \text{for } n = \pm 1, \pm 2, \ldots \\
t &= n\frac{\pi}{4} \qquad \text{for } n = \pm 1, \pm 2, \ldots.
\end{aligned}$$

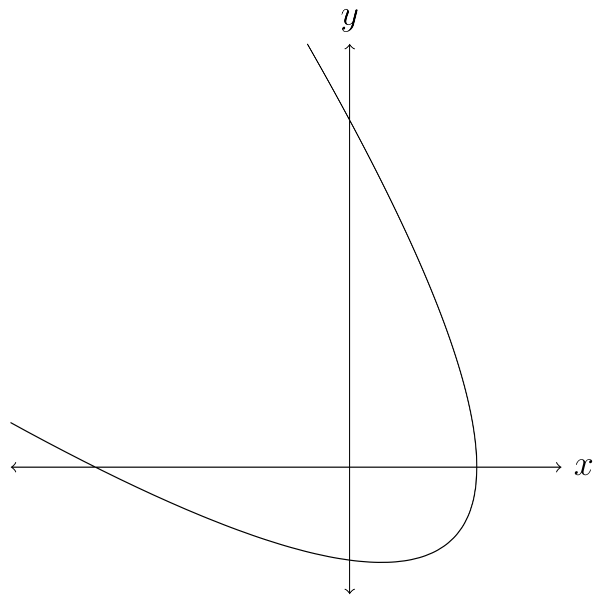
So vertical tangent lines occur at the points:

$$\begin{aligned}
\left(x\left(n\frac{\pi}{4}\right), y\left(n\frac{\pi}{4}\right)\right) &= \left(-\sin\left(2\cdot n\frac{\pi}{4}\right), \cos\left(2\cdot n\frac{\pi}{4}\right)\right) \\
&= \left(-\sin\left(n\frac{\pi}{2}\right), \cos\left(n\frac{\pi}{2}\right)\right) \\
&= (1,0) \text{ and } (-1,0).
\end{aligned}$$

Therefore, the vertical and horizontal tangent lines occur at points $(1,0)$, $(-1,0)$, $(0,1)$, and $(0,-1)$, which is what we predicted earlier.

Solutions should show all of your work, not just a single final answer.

Pictured below is the parametric curve $(x, y) = (3 - t^2, t^2 + 3t)$. It is a rotated parabola.



2. Mark the orientation on the curve (direction of increasing values of t).

3. Determine dy/dx in terms of the parameter t.

4. Find the slope of the tangent line at the point on the curve where it crosses the positive y-axis.

5. Find the point (x, y) on the curve where the tangent line is horizontal. (First, as a reality check, see which quadrant your answer should be in.)

6. T/F (with justification)

   On the parametric curve $(x, y) = (t^2 - 2t, t^3 - 3)$ the graph is increasing at the point where $t = 1/2$.