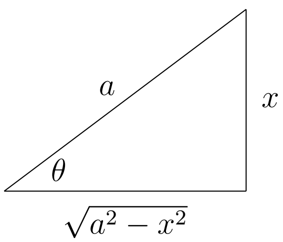
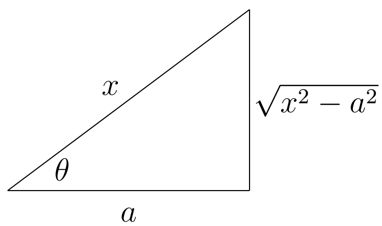
7.3 Trigonometric Substitution

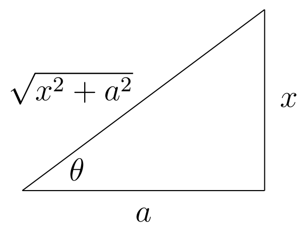
In each of the following trigonometric substitution problems, draw a triangle and label an angle and all three sides corresponding to the trigonometric substitution you select.

Summary of Trigonometric Substitution (understand how to make the triangles).

$\sqrt{a^2 - x^2}$	$\sqrt{x^2 - a^2}$	$\sqrt{x^2 + a^2}$





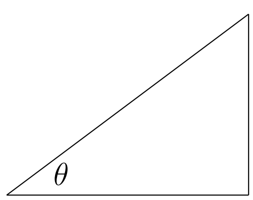


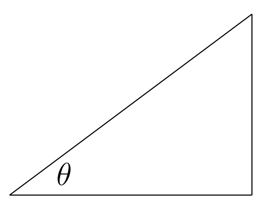
$\sin\theta = \dfrac{x}{a}, \quad -\dfrac{\pi}{2} \le \theta \le \dfrac{\pi}{2}$

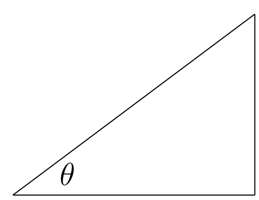
$\cos\theta = \dfrac{a}{x},$ that is, ________

$\tan\theta = \dfrac{x}{a},$ where

OR *OR* *OR*







________ $= \dfrac{x}{a},$ where

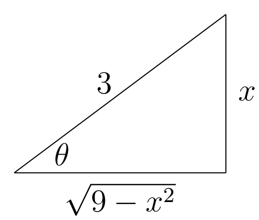
________ $= \dfrac{a}{x},$ that is, ________

________ $= \dfrac{x}{a},$ where

Example A (indefinite integral): Evaluate $\int \frac{dx}{\sqrt{9-x^2}}$.

Thinking about the problem:

Since the integrand involves $\sqrt{9-x^2}$ and there is not an extra factor of x in the numerator (if there were it might be possible to do a u-substitution with $u = 9 - x^2$), we will try a trigonometric substitution corresponding to a right triangle with a leg of length $\sqrt{9-x^2}$, hypotenuse 3, and the other leg has length x.



Doing the problem: Using the triangle diagram above,

$$\sin\theta = \frac{x}{3} \quad \text{where} \quad -\frac{\pi}{2} \le \theta \le \frac{\pi}{2},$$

so $x = 3\sin\theta$. Then $dx = 3\cos\theta\, d\theta$. Also from the triangle $\cos\theta = \frac{\sqrt{9-x^2}}{3}$, so $\sqrt{9-x^2} = 3\cos\theta$. The integral becomes

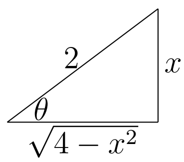
$$\begin{aligned}\int \frac{dx}{\sqrt{9-x^2}} &= \int \frac{3\cos\theta\, d\theta}{3\cos\theta} \\ &= \int d\theta \\ &= \theta + C.\end{aligned}$$

Since the substitution we used was $x = 3\sin\theta$, $\theta = \arcsin\left(\frac{x}{3}\right)$. So

$$\int \frac{dx}{\sqrt{9-x^2}} = \arcsin\left(\frac{x}{3}\right) + C.$$

Example B (definite integral): Evaluate $\displaystyle\int_0^{\sqrt{2}} \frac{x^2}{\sqrt{4-x^2}}\,dx$.

Thinking about the problem: Since the integrand involves $\sqrt{4-x^2}$ and there is not just a factor of x in the numerator (otherwise we can do a u-substitution with $u = 4 - x^2$), we try a trigonometric substitution with a triangle with a side of length $\sqrt{4-x^2}$, hypotenuse 2, and the other side has length x.



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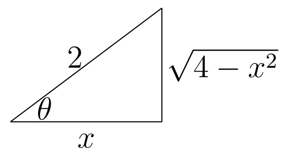
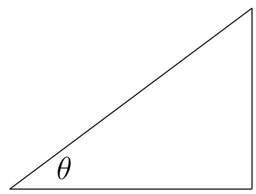


Doing the problem:

Solutions should show all of your work, not just a single final answer.

1. Evaluate $\int \frac{dx}{(9+x^2)^{3/2}}$.

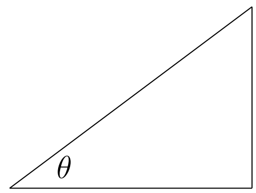
   (a) Fill in the sides of the right triangle below where $\sqrt{9+x^2}$ is one of the sides.

   (b) Using the sides of the triangle in (a), compute the indefinite integral. Write the final answer in terms of x.

2. Evaluate $\int \frac{\sqrt{x^2 - 9}}{x^3}\, dx$.

   (a) Fill in the sides of the right triangle below where $\sqrt{x^2 - 9}$ is one of the sides.

   (b) Using the sides of the triangle in (a), compute the indefinite integral. Write the final answer in terms of x.

3. Evaluate the definite integral $\int_0^3 \frac{x^2}{\sqrt{9-x^2}}\,dx$. (*Tip:* When you make a trigonometric substitution, change the bounds of integration as part of the substitution. See Example B.)

4. T/F (with justification): To evaluate $\int \frac{dx}{x^2\sqrt{x^2+2}}$ by trigonometric substitution, use $x = 2\tan\theta$.