7.1 Integration by Parts

Integration By Parts. Reverse *product rule*

$$u\,v' + v\,u' = \frac{d}{dx}(uv) \quad ,$$

and put $v' = \dfrac{dv}{dx}$ and $u' = \dfrac{du}{dx}$ to get *integration by parts.*

For indefinite integrals,

$$\int u\,dv + \int v\,du = uv \quad ,$$

where u and v are both functions of x, so

$$\boxed{\int u\,dv = uv - \int v\,du}.$$

For definite integrals,

$$\int_a^b u\,dv + \int_a^b v\,du = \left. uv \right|_a^b \quad ,$$

so

$$\boxed{\int_a^b u\,dv = \left. uv \right|_a^b - \int_a^b v\,du}.$$

Example A: Evaluate $\displaystyle\int x^2 e^x\,dx$.

Thinking about the problem:

Since the integral is a product of "unrelated" functions x^2 and e^x, we will use integration by parts. We have to pick u and dv, and then du and v. Passing from u to du is differentiation, and for any u in practice we can get du. However, passing from dv to v involves integration and that can be tricky. *The key is to pick dv first*: let dv be the most "complicated" expression in the integrand that you already know how to integrate, and then u is just the rest of the integrand. That is more practical advice than acronym-based tricks you may read elsewhere.

Doing the problem:

To figure out $\int x^2 e^x\,dx$ with integration by parts, the function e^x is more complicated (more sophisticated) than x^2. Set $dv = e^x\,dx$ and then $u = x^2$, so $u\,dv = x^2 e^x\,dx$

Next, we compute $du = 2x\,dx$ and $v = e^x$. Here are the results in a table.

$u = x^2$	$dv = e^x\,dx$
$du = 2x\,dx$	$v = e^x$

From $\int u\,dv = u\,v - \int v\,du$ we get

$$\int x^2 e^x\,dx = \int u\,dv = uv - \int v\,du = x^2 e^x - \int 2xe^x\,dx = x^2 e^x - 2\int xe^x\,dx.$$

Now we integrate by parts $\int xe^x\,dx$, which is simpler than what we started with. Again, e^x is the most complicated factor in the integrand that we know how to integrate. Set $dv = e^x\,dx$ and then $u = x$, so $u\,dv = xe^x\,dx$. We get $du = dx$ and $v = e^x$. as in the following table.

$u = x$	$dv = e^x\,dx$
$du = dx$	$v = e^x$

Then

$$\int xe^x\,dx = \int u\,dv = uv - \int v\,du = xe^x - \int e^x\,dx = xe^x - e^x + C.$$

Putting this all together,

$$\begin{aligned}\int x^2 e^x\,dx &= x^2 e^x - 2\int xe^x\,dx \\ &= x^2 e^x - 2\left(xe^x - e^x + C\right) \\ &= x^2 e^x - 2xe^x + 2e^x + 2C \\ &= x^2 e^x - 2xe^x + 2e^x + 2C.\end{aligned}$$

Since C is an arbitrary constant, we could change $2C$ to C in the last formula if we wished.

Example B: Evaluate $\int \tan^2 x \sec x \, dx$.

1. We want to evaluate $\int xe^{-3x}\,dx$.

   (a) Fill in the following table. (Hint: pick dv first, then u.)

$u =$	$dv =$
$du =$	$v =$

   (b) Evaluate the integral using integration by parts.

2. We want to evaluate $\int_0^\pi x\cos(3x)\,dx$.

   (a) Fill in the following table. (Hint: pick dv first, then u.)

$u =$	$dv =$
$du =$	$v =$

   (b) Evaluate $\int x\cos(3x)dx$ using integration by parts.

   (c) Using part (b), evaluate $\int_0^\pi x\cos(3x)dx$.

3. Evaluate $\int_0^\pi x^2 \sin x\,dx$.

4. Evaluate $\int e^x \cos(3x)\,dx$.

5. In 'In-class work 7.2' you found $\int \cos^2 x\,dx$ by the trick of writing $\cos^2 x$ in terms of $\cos(2x)$. Instead compute this with integration by parts. (Hint: $\sin^2 x + \cos^2 x = 1$.)

6. T/F (with justification): For differentiable $f(x)$, $\int_0^\pi f(x)\cos x\,dx = -\int_0^\pi f'(x)\sin x\,dx$.