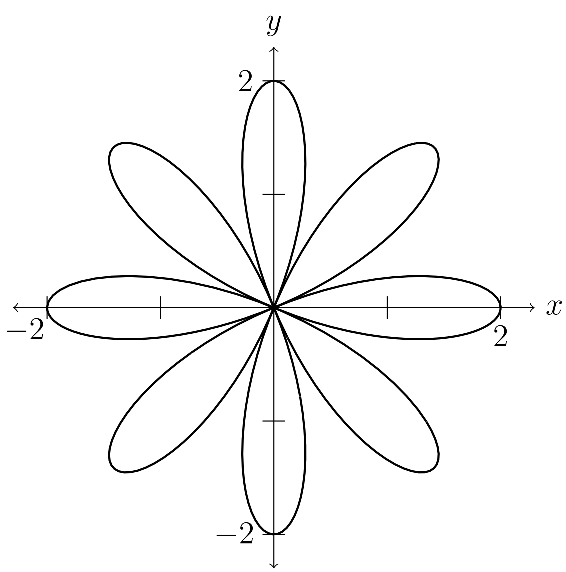
10.4 Areas in Polar Coordinates

Area. The area of a polar region bounded by $r = f(\theta)$ with $a \leq \theta \leq b$ and the lines $\theta = a$ and $\theta = b$ is $\displaystyle\int_a^b \frac{1}{2}r^2\,d\theta = \int_a^b \frac{1}{2}(f(\theta))^2\,d\theta$.

The area of the region bounded by two polar equations $r = f(\theta)$ and $r = g(\theta)$ for $a \leq \theta \leq b$, where $g(\theta) \leq f(\theta)$, is $\displaystyle\int_a^b \frac{1}{2}(f(\theta))^2\,d\theta - \int_a^b \frac{1}{2}(g(\theta))^2\,d\theta$. This is the polar analogue of the formula for the area between curves in Calculus I. As in Calculus I, if the polar curves cross then break up the region into parts where they don't cross.

Note: That a point has many polar coordinates means you need to carefully check, when setting up a polar area integral, that the chosen range of θ-values encloses the intended region and does so *just once*. Look closely at the graph when selecting the endpoints.

Example: Below is a graph of $r = 2\cos 4\theta$. Determine the area enclosed by it.

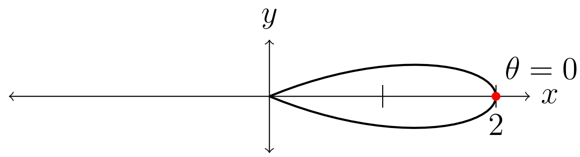


Thinking about the problem:

The graph has 8 petals, all with the same area, so the total area is 8 times the area of one petal. We will compute the area of one petal using the polar region area formula.

Doing the problem:

On the curve, when $\theta = 0$ we have $r = 2$, and $(r, \theta) = (2, 0)$ is the rightmost point on the petal crossing the positive x-axis. We will find the area of this petal (see figure below).



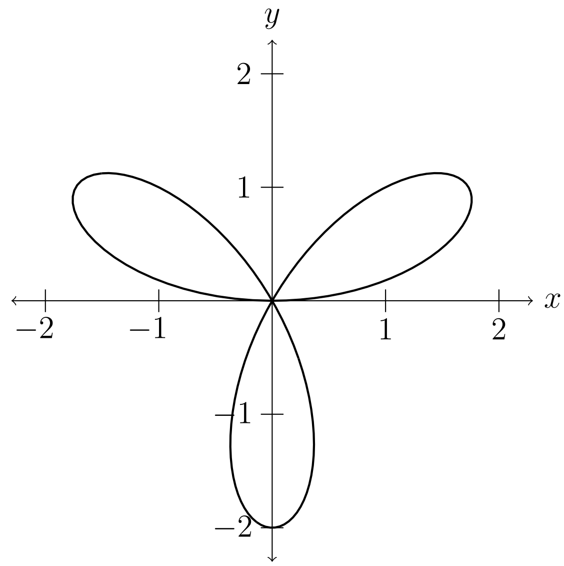
The graph is at the origin for the first time with $\theta > 0$ when $2\cos(4\theta) = 0$ for the smallest $\theta > 0$. That means $4\theta = \pi/2$, so $\theta = \pi/8$. By symmetry, the petal containing $(2, 0)$ is traced out for the continuous range of angles $-\pi/8 \le \theta \le \pi/8$. (While $-\pi/8 = 15\pi/8$ as polar angles, the graph for $\pi/8 \le \theta \le 15\pi/8$ is **not** the petal above, but all the others! Do you see why?) The area of the petal above is therefore

$$\begin{aligned}
\int_{-\pi/8}^{\pi/8} \frac{1}{2}(2\cos 4\theta)^2\, d\theta &= \int_0^{\pi/8} (2\cos 4\theta)^2\, d\theta \quad \text{since } \int_{-a}^{a} f(x)\, dx = 2\int_0^a f(x)\, dx \text{ for even } f(x) \\
&= \int_0^{\pi/8} 4\cos^2(4\theta)\, d\theta \\
&= \int_0^{\pi/8} 4 \cdot \frac{1+\cos(8\theta)}{2}\, d\theta \quad \text{since } \cos^2 x = \frac{1+\cos(2x)}{2} \\
&= \int_0^{\pi/8} (2 + 2\cos(8\theta))\, d\theta \\
&= \left(2\theta + \frac{1}{4}\sin(8\theta)\right)\Bigg|_0^{\pi/8} \\
&= \left(\frac{2\pi}{8} + \frac{1}{4}\sin(\pi)\right) - \left(0 + \frac{1}{4}\sin(0)\right) \\
&= \frac{\pi}{4} \quad \text{since } \sin(\pi) = 0.
\end{aligned}$$

Therefore the area enclosed by the whole graph (8 petals) is $8(\pi/4) = \boxed{2\pi}$.

Solutions should show all of your work, not just a single final answer.

1. Below is the graph of $r = 2\sin(3\theta)$, a 3-leaf rose.



(a) Show the point on the rose where $\theta = \pi/6$ is at the farthest distance from the origin by using the second derivative test for $r(\theta) = 2\sin(3\theta)$.

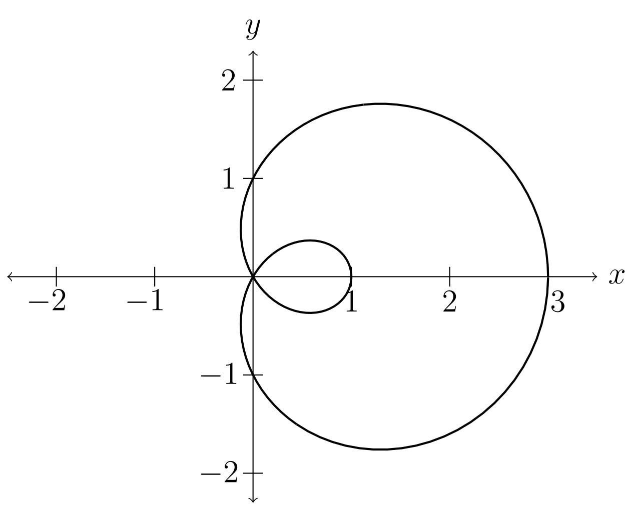
(b) Fill in the table below, mark the corresponding points on the rose, and draw arrows on each leaf to indicate the direction in which it is traced out as θ increases.

θ	0	$\pi/6$	$\pi/4$	$\pi/2$	$3\pi/4$	π
$r = 2\sin(3\theta)$						

(c) Determine the smallest positive angle θ at which the rose passes through the origin, and use this to help you set up an integral for the area of the leaf in the first quadrant. Be sure the bounds of integration are correct!

(d) Compute the integral in part c. (Hint: Recall from Section 7.2 how to integrate $\sin^2 u$ by writing it in terms of $\cos(2u)$.)

2. Below is the graph of $r = 1 + 2\cos\theta$. It is called a limaçon (French for "snail").



(a) Fill in the table below, mark the corresponding points on the rose, and draw arrows on the curve (including on both the big and small loops) to indicate the direction in which it is traced out as θ increases.

θ	0	$\pi/2$	π	$3\pi/2$	2π
$r = 1 + 2\cos\theta$					

(b) Determine all θ between 0 and 2π where the graph passes through the origin.

(c) Set up, **but do not evaluate**, an integral in terms of θ for the area enclosed by the inner loop of the limaçon. Be sure the bounds of integration are correct!

3. Set up, **but do not evaluate**, an integral in terms of θ for the area of the region that lies inside the circle $r = 3\sin\theta$ and outside the cardioid $r = 1 + \sin\theta$. Start by drawing a picture of these curves. (You can use `https://www.desmos.com/calculator` to see accurate graphs of these polar equations. Type "theta" in the equation box to get a θ.)

4. T/F (with justification)

   If the polar curve $r = f(\theta)$ for $a \le \theta \le b$ completely encloses a region, the area of the region is $\displaystyle\int_a^b \frac{1}{2} f(\theta)^2 \, d\theta$.