# Math 1152Q Problem Set A    Spring 2018

Please inform me of typos

Week 5 Monday, Feb 12

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(i) You are encouraged to discuss your work with your classmates but this assignment is to be completed individually (i.e, don't show anyone your full solutions). (ii) Check your work with other methods or with the computer whenever possible. (iii) After you've done some work, you may to ask for more hints during class, office hours, or on Piazza.

1. (Credit) This cannot be left blank. If you didn't talk to anyone or read other sources, please give an explanation.

   Please give credit to people that you talked with (including the people that *you* helped), any textbooks you used (other than Stewart), and the website addresses of any resources you have used.

2. (Style points) You will earn full "Style points" if each of your solution is legible, coherent, and not ambiguous. Your reader should not need to reread your solution several times to find a train of thought. Your final draft should not include any scratch work. In addition, you should use correct mathematical notations. This includes not writing an equal sign between two unequal objects and not treating the symbol $\infty$ like a number (for example, don't attempt to multiply 0 with the symbol $\infty$).

3. (Key words: Sec 11.2 series)

   i. There is a set $C$ which is formed by repeating the following process infinitely many times. Start with the interval $C_0 = [0, 1] = \{x \text{ a real number} : 0 \leq x \leq 1\}$. Remove the open middle third of this interval. Now you are left with $C_1 = [0, 1/3] \cup [2/3, 1]$. Next, remove the open middle thirds of each of the remaining intervals, i.e. remove $(1/9, 2/9)$ and $(7/9, 8/9)$. You are left with $C_2 = [0, 1/9] \cup [2/9, 1/3] \cup [2/3, 7/9] \cup [8/9, 1]$.

   The set $C$ is what you get after you by taking $C_n$ as $n \to \infty$.

   (a) Draw a separate number line representing each of $C_0, C_1, C_2$ and $C_3$. It will help you visualize what is going on if you draw the number line for $C_1$ below the number line for $C_0$ and so on.

   (b) What are the lengths of $C_0, C_1, C_2$ and $C_3$?

   (c) Find an expression for the length of $C_n$.

   (d) What are the lengths of the $C$? (Hint: use a series whose sum we can compute.)

   (e) Is $C$ empty? If not, list a few numbers that belong to $C$. If $C$ is empty, please explain.

ii. A two-dimensional counterpart of the above set is a 'carpet' square $S$ which is formed by repeating the following process infinitely many times. Start with a filled square of side 1, call it $S_0$. Remove the center square (of side 1/3) of the original square and now you are left with $S_1$. Then remove the center squares (of side 1/9) of the eight smaller remaining squares to get $S_2$, and so on.

The set $S$ is what you get by considering $S_n$ as $n \to \infty$

(a) Sketch $S_0, S_1$, and $S_2$ on the $x - y$ coordinate plane. Sketching $S_3$ is optional (lots of missing squares) but may be helpful.

(b) What are the areas of $S_0, S_1, S_2$ and $S_3$?

(c) Find an expression for the area of $S_n$.

(d) What is the area of the $S$? (Hint: use a series whose sum we know how to compute)

(e) Is $S$ empty? If not, list some points that belongs to $S$. If $S$ is empty, please explain.

4. (Key words: Sec 11.2 geometric series, decimal expansion)
   (a) Use geometric series to write $2.74\bar{9} = 2.74999999\ldots$ as a fraction.

   (b) Write a *different* decimal expansion for $2.74999999\ldots$. (Hint: can you use your solution above?)

   (c) i. Find the (*exact*, not rounded up) repeating decimal expansions of $\frac{1}{7}, \frac{2}{7}, \frac{3}{7}, \frac{4}{7}, \frac{5}{7},$ and $\frac{6}{7}$. You should practice doing at least two of them without the help of a computer (hint: do long division), but you don't need to include it in your submission.
   ii. List them in a table and note the interesting pattern.

   iii. Choose just ONE the six fractions above and show that your repeating decimal expansion is indeed the same as the fraction. If you don't enjoy working with many digits, you can use a computing tool - make sure it gives you *exact* fractions, not approximations (for example, WolframAlpha would work).

5. (Key words: geometric series)

The federal government uses a principle called the *multiplier effect* to justify deficit spending. According to this theory, a single dollar of government spending might increase total economic output by more than 1 dollar: When money is spent on goods and services, those who receive the money also spend some of it. The people receiving the money will spend some of that, and so on. This chain reaction is called the *multiplier effect* in economics.

In a hypothetical isolated community, the process begins when the government spends $D$ dollars. The people who receive it spends a fraction $c$ of those $D$ dollars (that is, $Dc$ dollars). Those who receive the $Dc$ dollars spend a fraction $c$ of it (that is, $Dc^2$ dollars), and so on. The value $c$ is called the *marginal propensity to consume* in economics. Assume that $0 < c < 1$ because you only spend less than 100% of what you have received.

**Optional:** Watch the following two videos if you don't understand the questions (written below) OR/and if you are interested in economics.

Khan Academy's "Marginal Propensity to Consume" video (less-mathy version): `https://www.khanacademy.org/economics-finance-domain/macroeconomics/income-and-expenditure-topic/mpc-tutorial/v/mpc-and-multiplier`

Khan Academy's "Marginal Propensity to Consume" video (with proof of the formula for partial sum of geometric series): `https://www.khanacademy.org/economics-finance-domain/macroeconomics/income-and-expenditure-topic/mpc-tutorial/v/mathy-version-of-mpc-and-multi`

(a) Let $S_n$ be the total spending that has been generated after $n$ transactions. Find a formula for $S_n$. (Hint: see the explanation for eq. 3 on page 709 or watch the second video above).

(b) Suppose that each recipient of spent money saves $s$ of the money that they receive. This value $s$ is called the *marginal propensity to save* in economics. (Note: each person spends $c$ of the money the receive and saves $s$ of the money they receive, so $c + s = 1$). Show that $\lim_{n\to\infty} S_n = \left(\frac{1}{s}\right) D$. (Hint: see the solution to Sec 11.2, Example 2 on page 709)

(c) Let

$$k := \frac{1}{s}.$$

This number $k$ is called the *multiplier* in economics. What is the multiplier if the marginal propensity to consume is 60%? (Hint: watch the above video/s if you don't know how to compute the answer).

6. (Key words: harmonic series, divergence) Suppose you have a large supply of rectangles (like books or a deck of cards), all the same size, and you stack them at the edge of a table, with each book extending farther beyond the edge of the table than the one beneath it. Our goal is to show that it is possible to do this so that the top book or card can extend any distance at all beyond the edge of the table if the stack is high enough.

   Use the following method of stacking: the top book extends half its length beyond the second book. The second book extends a quarter of its length beyond the third. The third extends one-sixth of its length beyond the fourth, and so on. See `https://d3njjcbhbojbot.cloudfront.net/api/utilities/v1/imageproxy/https://coursera-course-photos.s3.amazonaws.com/37/d1a6e9c4c9f32a3` `self-describing-sequence-and-harmonic-series.png`.

   a.) (BONUS) For this activity, you are encouraged to get together with a classmate or two (but no more than three people total including yourself). Try this activity yourself with a deck of cards so that the top card extends beyond the bottom card. To receive the bonus points, post a picture of your work on Piazza (either publicly or privately). All the participants (no more than 3) should be in the picture, and the picture should show clearly that the top object extend beyond the end of your bottom object.

   b.) Afterwards, watch this Coursera video: "how far can you build a one-sided bridge?" `https://www.coursera.org/learn/advanced-calculus/lecture/9AUVI/how-far-out-can-you-build-a-one` (You don't need to sign up to view the video - just click exit whenever a pop-up appears - and you can watch at faster speed by clicking the setting button on the bottom right of the page).

(Con't from part b) Your job: Write a letter to a friend (who is also taking Calc II) explainig what the video is about. You must draw pictures (or insert pictures from elsewhere if you don't like to draw).

7. (Key words: vocabulary, limits and sequences, convergent series)

   (a) (copy from Sec 11.1 page 696) Let $\{a_n\}$ be a sequence and let $L \in \mathbb{R}$ (this notation means that $L$ is a real number). What does $\lim_{n\to\infty} a_n = L$ mean? Warning: do not include variations of the words "converge", "diverge", "approach", or "infinity" in your answer. Another warning: in general $\lim_{n\to\infty} a_n = L$ does *NOT* imply that "$a_n$ will never reach $L$".

   ______________________________________________________________.

   (b) (copy from Sec 11.2, page 708) Let $\{c_n\}$ be a sequence. What is a partial sum of $\{c_n\}$?

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   (c) (copy from Sec 11.2, page 708) Let $\{c_n\}_{n=1}^{\infty}$ be a sequence. We say that the infinite series $\sum_{n=1}^{\infty} c_n$ is *convergent* if

   ______________________________________________________________. (Hint: your answer should include the words 'limit' and 'partial sums')

8. (OPTIONAL PROBLEMS, Key words: Sec 11.1, precise definition) **These problems (taken from Exam 1 practice) are optional, but I will give you feedback if you submit them. Please circle two out of three, and submit polished answers of the circled sequences.** Each of the following series $a_k$ converges to some real number $L$. For $\epsilon > 0$, find $a$ positive number $N$ such that, if $k > N$, then $a_k$ is within distance $\epsilon$ of $L$.

   SEE EXAMPLE `https://egunawan.github.io/spring18/notes/notes11_1choosingN.pdf`

   (a) $a_k = \frac{1}{k^2+3}$, $L = 0$.

   (b) $a_k = \frac{3k+2}{2k-1}$, $L = \frac{3}{2}$.

   (c) $a_k = \frac{k^2+2}{k^2-3}$, $L = 1$.

9. (Key words: Sec 11.1, squeeze theorem for sequences) The sequence $a_n = (5n + 4)/(3n^2 - 2)$ converges to 0. Justify this fact by squeezing $a_n$ between 0 and another sequence of type $b_n =$ (a constant)$/n$ and using the Squeeze theorem. You may assume $\lim_{n\to\infty}$ (any constant)$/n = 0$.

   See a similar (model) solution at: `https://egunawan.github.io/spring18/hw/models.pdf` (Prob 3)

10. (Write your own problem) Review the past few weeks of topics from Calc II. The solution should not require topics beyond Calculus II materials. A few questions submitted may be chosen for future questions.

    (a) Explain briefly which concepts you wish to highlight and why (for example, maybe it's a brand-new concept to you or has applications to science).

    (b) Write a new problem. *(Do not write an answer key on the same page).* It should require a decent understanding of one or more of the concepts/ skills to solve. For ideas, you may look at Stewart's textbook problems or other textbooks or WebAssign. The problem's solution should require more than simply writing down a definition from the textbook.

    (c) i. Ask a couple Calc II students to solve your problem without showing them your answer. If they are having trouble solving it, explain your solution. Write the names of the people who have attempted to solve your problem.

    ii. Write a brief comment about what other people thought (ask them to give a comment more constructive than 'good job' or 'this is a difficult problem', etc). If their (correct) solution is different from your solution, please write their solution on the next page.

(Please go to the next page)

(d) Write a complete answer key to your problem.