Contents

1. (Limit laws and L'Hospital's Rule Sec 4.4)

i.) If $\lim_{x\to\infty} f(x) = 0$, then $\lim_{x\to\infty} (xf(x))$ is ...
(a) zero (b) ∞ (c) non-zero constant (d) another method is needed to determine this

Solution: Answer: $\boxed{\text{Not enough information}}$. For example, $\lim_{x\to\infty} x\left(\frac{1}{x^2}\right) = 0$, $\lim_{x\to\infty} x\left(\frac{1}{x}\right) = 1$, $\lim_{x\to\infty} x\sin\left(\frac{1}{x}\right) = 1$, $\lim_{x\to\infty} x\left(\frac{1}{\ln x}\right) = \infty$

ii.) If $\lim_{x\to\infty} f(x) = \infty$, then $\lim_{x\to\infty} (f(x) - x)$ is ...
(a) zero (b) ∞ (c) non-zero constant (d) another method is needed to determine this

Solution: Answer: $\boxed{\text{Not enough information}}$. For example, $\lim_{x\to\infty} x - x = 0$, $\lim_{x\to\infty} (x+1) - x = 1$, $\lim_{x\to\infty} x^2 - x = \infty$.

iii.) If $\lim_{x\to 0} f(x) = 0$, then $\lim_{x\to 0} (f(x) - x)$ is ...
(a) zero (b) ∞ (c) non-zero constant (d) another method is needed to determine this

Solution: Answer: $\boxed{\text{zero}}$.

iv.) If $\lim_{x\to 0} f(x) = \infty$ and $\lim_{x\to 0} g(x) = \infty$, then $\lim_{x\to 0} (f(x) + g(x))$ is ...
(a) zero (b) ∞ (c) non-zero constant (d) another method is needed to determine this

Solution: Answer: $\boxed{\infty}$

v.) If $\lim_{x\to\infty} f(x) = 1$, then $\lim_{x\to\infty} (f(x))^x$ is ...
(a) zero (b) ∞ (c) 1 (d) another method is needed to determine this.

Solution: Answer: $\boxed{\text{not enough information}}$. For example, $\lim_{x\to\infty} \left(1 + \frac{1}{x}\right)^x = e$ and $\lim_{x\to\infty} 1^x = 1$.

vi.) Evaluate $\lim_{x\to 1} \frac{\ln x}{x-1}$ and $\lim_{x\to\infty} \frac{e^x}{x^2}$.

Solution: $\boxed{1}$ and $\boxed{\infty}$. Sec 4.4 Example 1, pg 306 and Sec 4.4 Example 2, pg 306.

vii.) Evaluate $\lim_{x\to\infty} \frac{\ln x}{\sqrt{x}}$ and $\lim_{x\to 0} \frac{\tan x - x}{x^3}$.

Solution: $\boxed{0}$ and $\boxed{\frac{1}{3}}$. Sec 4.4 Example 3, pg 307 and Sec 4.4 Example 4, pg 307.

viii.) Evaluate $\lim_{x\to\pi^-} \dfrac{\sin x}{1-\cos x}$.

Solution: $\boxed{0}$. Sec 4.4 Example 5, pg 308.

ix.) Evaluate $\lim_{x\to 0^+} x\ln x$. Note: This shows up frequently when we compute our improper integral examples.

Solution: $\boxed{0}$. Sec 4.4 Example 6, pg 308.

x.) Evaluate $\lim_{x\to 0^+} x(\ln x)^3$. Note: This shows up frequently when we compute our improper integral examples.

Solution: $\boxed{0}$. Similar to Sec 4.4 Example 6, pg 308, but we apply L'hospital's Rule three times.

1 Part 2: 7.1-7.4 Integration Techniques, 7.8 Improper Integrals

2. (Various definite and indefinite integrals).

(a) Is $\int_1^2 t^3 \ln(t)dt$ a proper or improper integral?

Solution: $\int_1^2 t^3 \ln(t)dt$ is a $\boxed{\text{proper}}$ integral since neither t^3 nor $\ln(t)$ has discontinuity on the interval $[1,2]$.

(b) Evaluate $\int t^3 \ln(t)dt$ and $\int_1^2 t^3 \ln(t)dt$

Solution: Answer: $\boxed{\frac{1}{4}\left(t^4 \ln(t) - \frac{t^4}{4}\right) + C}$ and $\boxed{\ln(16) - \frac{15}{16}}$.

(c) Perform either a proof or a reality check for the previous problem. For example, differentiate your answer (for a proof) or check that your definite integral is a positive number, since $t^3 ln(t)$ is positive for $t > 1$.

(d) $\int \dfrac{2x-3}{8+x^2}\,\mathrm{dx}$

Solution: Answer: Draw a triangle and use inverse trig substitution. Then use u-substitution where $\boxed{u = \cos(\theta)}$ and $du = -\sin(\theta)$. Then you get $\int \dfrac{2x-3}{8+x^2}\,\mathrm{dx} = \boxed{ln(x^2+8) - \frac{3}{2\sqrt{2}}\arctan\left(\frac{x}{2\sqrt{2}}\right) + C}$.

(e) $\int \dfrac{\sin(\ln(x))}{x}\,\mathrm{dx}$

Solution: Answer: Use u-substitution with $u = \ln(x)$ and $du = \frac{1}{x}\,\mathrm{dx}$. Then $\int \dfrac{\sin(\ln(x))}{x}\,\mathrm{dx} = \boxed{-\cos(\ln(x)) + \text{Constant}}$.

(f) Is $\int_0^1 x\, e^{-x^2}$ a proper or improper integral?

Solution: $\int_0^1 x\, e^{-x^2}$ is a $\boxed{\text{proper}}$ integral: Both $-x^2$ and e^x is continuous on $[0,1]$, so e^{-x^2} is also continuous on $[0,1]$, since the composition of continuous functions is continuous (reference: Sec 2.5 Theorem 9, pg 121). Since x is continuous on $[0,1]$

and the product of continuous functions is continuous (reference: Sec 2.5 Theorem 4, pg 117), we conclude that $x\ e^{-x^2}$ is continuous on $[0,1]$ (and, in fact, everywhere - but this fact isn't relevant to this situation).

(g) Evaluate $\int_0^1 x\ e^{-x^2}$

Solution: Use u-sub with $\boxed{u=-x^2}$, $du = -2x$ dx. Then $\int_0^1 x\ e^{-x^2} = -\frac{1}{2}e^u\Big|_0^{-1} = \boxed{\frac{1}{2}(1-e^{-1})}$.

(h) Perform a reality check for your answer to the previous problem.

Solution: A possible reality check: your answer should be positive because xe^{-x^2} is positive on the interval $(0,1]$.

(i) Evaluate $\int (x+2)\sin(3x)$ dx

Solution: Answer: Integration by parts with $\boxed{u = x+2 \text{ and } dv = \sin(3x)\text{ dx}}$.

Then $\int(x+2)\sin(3x)$ dx $= \boxed{-\frac{1}{3}(x+2)\cos(3x) + \frac{1}{9}\sin(3x) + \text{Constant}}$.

(j) Prove that your previous answer is correct by differentiating and applying the Fundamental Theorem of Calculus.

(k) Evaluate $\int e^{\sqrt{x}}$ dx

Solution: Answer: Do substitution with $w=\sqrt{x}$ and $dw = \frac{1}{2}x^{-\frac{1}{2}}$ dx. Then do integration by parts with $u = w$ and $dv = e^w$ dx. Then $\int e^{\sqrt{x}}$ dx $= 2we^w - 2\int e^w dw = 2we^w - 2e^w = \boxed{2\sqrt{x}e^{\sqrt{x}} - 2e^{\sqrt{x}} + \text{C}}$.

(l) Prove that your previous answer is correct by differentiating and applying the Fundamental Theorem of Calculus.

(m) Evaluate $\int \frac{x}{\sqrt{4-x^2}}$

Solution: Answer: You can do trig sub, but u-substitution may be faster. $\int \frac{x}{\sqrt{4-x^2}} = -\frac{1}{2}\int \frac{1}{\sqrt{u}}du = -u^{1/2} + C =$ $\boxed{-\sqrt{4-x^2}+C}$.

(n) Prove that your previous answer is correct by differentiating and applying the Fundamental Theorem of Calculus.

(o) Is $\int_0^9 \frac{1}{\sqrt{x}}$ dx an improper integral?

Solution: Yes, $\lim_{x\to 0^+} \frac{1}{\sqrt{x}} = \infty$, so $\frac{1}{\sqrt{x}}$ has an infinite discontinuity at 0.

(p) Determine whether $\int_0^{25} \frac{1}{\sqrt{x}}$ dx or $\int_0^{16} \frac{1}{\sqrt{x}}$ dx or $\int_0^{9} \frac{1}{\sqrt{x}}$ dx is convergent or divergent. If it is convergent, evaluate the integral.

Solution: Answer: $\boxed{5*2}$ or $\boxed{4*2}$ or $\boxed{3*2}$.

(q) True of false? Let $a>0$. $\int_0^a f(x) = \int_0^a f(a-x)$ dx. **T** **F**
If T, justify. If F, give a counterexample.

Solution: Answer: $\boxed{\text{True}}$. (Hint: use u-substitution $u = a - x$.)

(r) True of false? Let $a > 0$. $\int_0^a f(x) = \int_0^a f(x-a)\ \mathrm{dx}$. **T** **F**
If T, justify. If F, give a counterexample.

Solution: Answer: $\boxed{\text{False}}$ in general. A possible counterexample: let $f(x) = x$. Then $\int_0^2 x\ \mathrm{dx} = 2$ but $\int_0^2 (x-2)\ \mathrm{dx} = -2$.
Another possible counterexample: let $f(x) = sin(x)$. Then $\int_0^\pi \sin(x)\ \mathrm{dx} = 2$ but $\int_0^\pi \sin(x-\pi)\ \mathrm{dx} = -2$. (Hint: use u-substitution $u = x - a$.)

(s) Write a sanity-check-type calculation (different from what you've written above) to further confirm your answer in the previous two questions.

3. (From class handouts)

(a) Is the integral $\int_0^\pi \sin^3(5x)\,dx$ a proper or improper integral?

Solution: This is a proper integral because $\sin(5x)$ is continuous on $[0, \pi]$, and so $(\sin(5x))^3$ is also continuous on $[0, \pi]$.

(b) Evaluate $\int_0^\pi \sin^3(5x)\,dx$.

Solution: *Thinking about the problem:*
To integrate a power like $\sin^3(5x)$, let's write $\sin^3\theta$ in terms of lower powers. By the first trigonometric identity above, we can write $\sin^2\theta = 1 - \cos^2\theta$, so

$$\sin^3\theta = \sin^2\theta\sin\theta = (1-\cos^2\theta)\sin\theta.$$

Therefore (using $\theta = 5x$)

$$\int_0^\pi \sin^3(5x)\,dx = \int_0^\pi (1-\cos^2(5x))\sin(5x)\,dx.$$

Doing the problem:
After rewriting of the function being integrated, let's use the substitution $u = \cos(5x)$, so $du = -5\sin(5x)\,dx$:

$$\int (1-\cos^2(5x))\sin(5x)\,dx = \int (1-u^2)\frac{-du}{5} = -\frac{1}{5}\int (1-u^2)\,du.$$

Let's turn x-bounds into u-bounds in the definite integral:

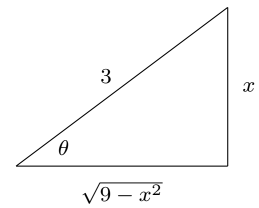
$$x = 0 \Longrightarrow u = \cos(5\cdot 0) = \cos 0 = 1, \quad x = \pi \Longrightarrow u = \cos(5\pi) = -1.$$

Therefore

$$\begin{aligned}
\int_0^\pi \sin^3(5x)\,dx &= \int_{x=0}^{x=\pi} (1-\cos^2(5x))\sin(5x)\,dx \\
&= -\frac{1}{5}\int_{u=1}^{u=-1} (1-u^2)\,du \quad \text{(Note the order of integration)} \\
&= \frac{1}{5}\int_{-1}^{1} (1-u^2)\,du \quad \text{(Sign change in the order of integration)} \\
&= \frac{1}{5}\left(u - \frac{u^3}{3}\right)\Big|_{-1}^{1} \\
&= \frac{1}{5}\left(\left(1-\frac{1}{3}\right) - \left(-1-\frac{-1}{3}\right)\right) \\
&= \frac{1}{5}\left(1-\frac{1}{3}+1-\frac{1}{3}\right) \\
&= \frac{1}{5}\left(2-\frac{2}{3}\right) \\
&= \boxed{\frac{4}{15}}.
\end{aligned}$$

(c) Evaluate $\displaystyle\int \frac{dx}{\sqrt{9-x^2}}$.

Solution: *Thinking about the problem:*
Since the integrand involves $\sqrt{9-x^2}$ and there is not an extra factor of x in the numerator (if there were it might be possible to do a u-substitution with $u = 9 - x^2$), we will try a trigonometric substitution corresponding to a right triangle with a leg of length $\sqrt{9-x^2}$, hypotenuse 3, and the other leg has length x.



Doing the problem: Using the triangle diagram above,

$$\sin\theta = \frac{x}{3} \quad \text{where } -\frac{\pi}{2} \le \theta \le \frac{\pi}{2},$$

so $x = 3\sin\theta$. Then $dx = 3\cos\theta\, d\theta$. Also from the triangle $\cos\theta = \dfrac{\sqrt{9-x^2}}{3}$, so $\sqrt{9-x^2} = 3\cos\theta$. The integral becomes

$$\begin{aligned}\int \frac{dx}{\sqrt{9-x^2}} &= \int \frac{3\cos\theta\, d\theta}{3\cos\theta}\\ &= \int d\theta\\ &= \theta + C.\end{aligned}$$

Since the substitution we used was $x = 3\sin\theta$, $\theta = \arcsin\left(\frac{x}{3}\right)$. So

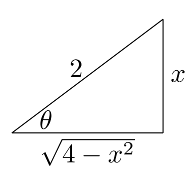
$$\int \frac{dx}{\sqrt{9-x^2}} = \boxed{\arcsin\left(\frac{x}{3}\right) + C}.$$

(d) Consider $\displaystyle\int_0^{\sqrt{2}} \frac{x^2}{\sqrt{4-x^2}}dx$, $\displaystyle\int_{-\sqrt{2}}^{0} \frac{x^2}{\sqrt{4-x^2}}dx$, $\displaystyle\int_0^{2} \frac{x^2}{\sqrt{4-x^2}}dx$ and $\displaystyle\int_{-2}^{\sqrt{2}} \frac{x^2}{\sqrt{4-x^2}}dx$. Which are proper integrals and which are improper integrals?

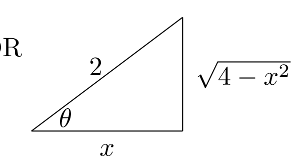
Solution: The function $\dfrac{x^2}{\sqrt{4-x^2}}$ is continuous on $[-\sqrt{2}, \sqrt{2}]$, so $\displaystyle\int_0^{\sqrt{2}} \frac{x^2}{\sqrt{4-x^2}}dx$ and $\displaystyle\int_{-\sqrt{2}}^{0} \frac{x^2}{\sqrt{4-x^2}}dx$ are both proper integrals.
The integrals $\displaystyle\int_0^{2} \frac{x^2}{\sqrt{4-x^2}}dx$ and $\displaystyle\int_{-2}^{\sqrt{2}} \frac{x^2}{\sqrt{4-x^2}}dx$ are both improper integrals because the function $\dfrac{x^2}{\sqrt{4-x^2}}$ has infinite discontinuities at $x=-2$ and $x=2$. ($\displaystyle\lim_{x\to 2^-} \frac{x^2}{\sqrt{4-x^2}} = \infty$ and $\displaystyle\lim_{x\to 2^+} \frac{x^2}{\sqrt{4-x^2}} = \infty$).

(e) Evaluate $\displaystyle\int_0^{\sqrt{2}} \frac{x^2}{\sqrt{4-x^2}}dx$.

Solution: *Thinking about the problem:* Since the integrand involves $\sqrt{4-x^2}$ and there is not just a factor of x in the numerator (otherwise we can do a u-substitution with $u = 4 - x^2$), we try a trigonometric substitution with a triangle with a side of length $\sqrt{4-x^2}$, hypotenuse 2, and the other side has length x.



OR



Doing the problem: See Example B from Week 6 day 2 Section 7.3 notes:
`https://www.dropbox.com/s/l0cwr7rtly8xncp/week6d2_worksheet7_3trig_sub_pages1_and_3.pdf?dl=0`

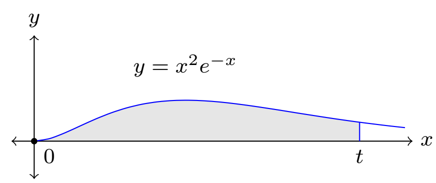
(f) Without explicitly trying to evaluate this integral, determine whether it is possible for $\int_0^\infty x^2 e^{-x}\,dx$ to be convergent to a negative value.

Solution: It is not possible because $x^2 e^{-x} \geq 0$ for all $x \geq 0$. See sketch in the solution of the next part.

(g) Is $\int_0^\infty x^2 e^{-x}\,dx$ convergent or divergent? If convergent, evaluate it.

Solution: *Thinking about the problem:*

The integral is $\lim_{t\to\infty} \int_0^t x^2 e^{-x}\,dx$ and the graph of $y = x^2 e^{-x}$ is below. We will compute $\int_0^t x^2 e^{-x}\,dx$ and see how it behaves as $t \to \infty$.



To evaluate $\int_0^t x^2 e^{-x}\,dx$ we will use integration by parts.

Doing the problem:

To work out $\int_0^t x^2 e^{-x}\,dx$ with integration by parts set u and dv to be as in the chart below, and then compute du and v.

$u = x^2$	$dv = e^{-x}\,dx$
$du = 2x\,dx$	$v = -e^{-x}$

Thus $\int_0^t x^2 e^{-x}\,dx = uv\Big|_0^t - \int_0^t v\,du = -x^2 e^{-x}\Big|_0^t + \int_0^t 2xe^{-x}\,dx = -\frac{t^2}{e^t} + 2\int_0^t xe^{-x}\,dx$. We work out the new integral also using integration by parts, starting with the chart below.

$u = x$	$dv = e^{-x}\,dx$
$du = dx$	$v = -e^{-x}$

Thus

$$\int_0^t xe^{-x}\,dx = -xe^{-x}\Big|_0^t + \int_0^t e^{-x}\,dx = -\frac{t}{e^t} - e^{-x}\Big|_0^t = -\frac{t}{e^t} - \frac{1}{e^t} + 1,$$

so returning to the initial calculation we have

$$\int_0^t x^2 e^{-x}\,dx = -\frac{t^2}{e^t} + 2\int_0^t xe^{-x}\,dx = -\frac{t^2}{e^t} + 2\left(-\frac{t}{e^t} - \frac{1}{e^t} + 1\right) = -\frac{t^2}{e^t} - \frac{2t}{e^t} - \frac{2}{e^t} + 2.$$

Letting $t \to \infty$, $\int_0^\infty x^2 e^{-x}\,dx = \lim_{t\to\infty}\left(-\frac{t^2}{e^t} - \frac{2t}{e^t} - \frac{2}{e^t} + 2\right) = 0 - 0 - 0 + 2$ by L'Hospital's rule (used twice for the first expression). Thus $\int_0^\infty x^2 e^{-x}\,dx = 2$: the improper integral is convergent and equals 2.

(h) (Note: You may use the fact sheets to look up derivatives and integrals of trig functions)

Evaluate 1.) $\int_0^{\frac{\pi}{2}} \sin^5 x\ \mathrm{dx}$ or 2.) $\int \frac{\cos^5 x}{\sin x}\ \mathrm{dx}$ or 3.) $\int_0^\pi \cos^4(2x)\ \mathrm{dx}$ or 4.) $\int \sin^3 x \cos^5 x\ \mathrm{dx}$ or 5.) $\int \sin^2 x \cos^2 x\ \mathrm{dx}$ or 6.) $\int \tan^3 x \sec^3 x\ \mathrm{dx}$ or 7.) $\int \tan^2 x \sec^4 x\ \mathrm{dx}$ or 8.) $\int_0^{\frac{\pi}{4}} \sec^4 x\ \mathrm{dx}$ or 9.) omitted or 10.) $\int \tan^3 x \sec^4 x\ \mathrm{dx}$.

Solution: `https://egunawan.github.io/spring18/notes/hw7_2key.pdf`

(i) Evaluate 1.) $\int \frac{1}{x\sqrt{4-x^2}}\ \mathrm{dx}$ or 2.) $\int \frac{1}{\sqrt{x^2+16}}\ \mathrm{dx}$ or 3.) $\int_{\sqrt{2}}^2 \left(\frac{1}{x^3\sqrt{x^2-1}}\right)\ \mathrm{dx}$ or 4.) $\int_0^{\frac{\pi}{2}} \frac{\cos x}{\sqrt{1+\sin^2 x}}\ \mathrm{dx}$ or 5.) $\int_0^1 \left(\frac{1}{\sqrt{-x^2+2x+3}}\right)\ \mathrm{dx}$ (Hint: First complete the square) or 6.) $\int \frac{1}{\sqrt{1+16x^2}}\ \mathrm{dx}$ or 7.) $\int_0^{\sqrt{2}} \frac{x^2}{\sqrt{4-x^2}}\ \mathrm{dx}$ or 8.) $\int \frac{1}{x^2\sqrt{9x^2-1}}\ \mathrm{dx}$ or

9.) $\int \sqrt{5+4x-x^2}$ dx. (Hint: First complete the square)

Solution: `https://egunawan.github.io/spring18/notes/hw7_23key.pdf`

(j) 1.) Evaluate $\int \ln\left(x+\sqrt{1+x^2}\right)$ dx or 2.) $\int x \tan^2 x$ dx or 3.) $\int \cos(\sqrt{x})$ dx or
4.) Evaluate $\int x^2 (\ln x)^2$ dx or 5.) omitted or 6.) $\int \cos(\ln x)$ dx or
7a.) How can you derive the formula for Integration by Parts? or
7b.) Evaluate $\int_0^{\frac{\pi}{2}} x \cos(2x)$ dx or
7c.) Suppose $f(1)=2$, $f(4)=7$, $f'(1)=5$, $f'(4)=3$. Suppose f'' is continuous. Evaluate $\int_1^4 x\, f''(x)$ dx
or 7d.) Evaluate $\int \arctan x$ dx or 7e.) $\int e^x \cos x$ dx or
7f.) A particle that moves along a straight line has velocity $v(t)=t^3e^{-t}$ meters per second after t seconds. How far will it travel during the first t seconds?

Solution: `https://egunawan.github.io/spring18/notes/hw7_1key.pdf`

(k) Evaluate 1.) $\int_0^\infty e^{-2x}$ dx or 2.) $\int_1^\infty \frac{1}{\sqrt{x}}$ or 3.) $\int_1^\infty \sin^2 x$ dx or 4.) $\int_1^\infty \frac{1}{x^2+2x-3}$ dx.

Solution: `https://egunawan.github.io/spring18/notes/LA7_8part1key.pdf`

(l) 1.) Evaluate $\int_0^8 \frac{1}{\sqrt[3]{x}}$ dx or 2.) Evaluate $\int_1^{\frac{\pi}{2}} \sec x$ or 3.) Evaluate $\int_0^5 \frac{x}{x-2}$ dx or
4.) Use the Comparison Theorem to determine whether $\int_1^\infty \frac{x}{x^3+1}$ dx.

Solution: `https://egunawan.github.io/spring18/notes/LA7_8part2key.pdf`

4. (Integrating rational functions)

(a) Evaluate $\int \frac{2x+1}{x^2-4}\,dx$.

Solution: Note: It is also possible to evaluate this using the trig substitution method, but the following will walk you through the Partial Fraction Decomposition method.

Thinking about the problem:

The integrand $\frac{2x+1}{x^2-4}$ is a rational function and does not look like it can be handled with substitution, so we use partial fractions. The denominator x^2-4 is $(x+2)(x-2)$, a product of different linear factors, so the partial fraction decomposition of $\frac{2x+1}{x^2-4}$ is $\frac{A}{x+2}+\frac{B}{x-2}$ for some constants A and B. After solving for A and B we would have $\int \frac{2x+1}{x^2-4}\,dx = \int \frac{A}{x+2}\,dx + \int \frac{B}{x-2}\,dx$ and can integrate the right side.

Doing the Problem:

Writing $\frac{2x+1}{x^2-4} = \frac{A}{x+2}+\frac{B}{x-2}$, solve for A and B by multiplying both sides by the denominator x^2-4:

$$2x+1 = A(x-2)+B(x+2).$$

Setting $x=2$, we find

$$2(2)+1 = 5 = A(0)+B(2+2) = 4B \Rightarrow 5 = 4B \Rightarrow B = \frac{5}{4}.$$

Setting $x=-2$, we find

$$2(-2)+1 = A(-2-2)+B(0) \Rightarrow -3 = -4A \Rightarrow A = \frac{3}{4}.$$

Therefore $\frac{2x+1}{x^2-4} = \frac{3/4}{x+2} + \frac{5/4}{x-2}$, so

$$\int \frac{2x+1}{x^2-4}\,dx = \frac{3}{4}\int \frac{dx}{x+2} + \frac{5}{4}\int \frac{dx}{x-2}$$
$$= \boxed{\frac{3}{4}\ln|x+2| + \frac{5}{4}\ln|x-2| + C}.$$

(b) Decompose $\frac{3x^2+2x-3}{x^3-x}$ into partial fractions.

(c) Provide a computation that is either a formal verification (that is, a proof) or simply a reality-check for your answer to the previous question.

Solution: For example, you can rewrite your result as one fraction and check that it's equal to $\frac{3x^2+2x-3}{x^3-x}$

(d) Evaluate $\int \frac{10}{(x+5)(x-2)}$ dx.

Solution: Answer: $\boxed{\frac{10}{7}(\ln(x-2) - \ln(5+x))} + C$

(e) Provide a computation that is either a formal verification (that is, a proof) or simply a sanity-check for your answer to the previous question.

Solution: For example, you can check that the derivative of your result is equal to $\frac{10}{(x+5)(x-2)}$.

(f) Evaluate $\int \frac{9}{(x-6)(x+3)}$ dx or $\int \frac{12}{(x-2)(x+1)}$ dx or $\int \frac{8}{(x-1)(x+3)}$ dx.

Solution: Answer: After applying partial fraction decomposition,
the integrand is equal to $\frac{1}{x-6} - \frac{1}{x+3}$ or $\frac{4}{x-2} - \frac{4}{x+1}$ or $\frac{2}{x-1} - \frac{2}{x+3}$.
The antiderivative is $\boxed{\ln\left|\frac{x-6}{x+3}\right| + C}$ or $\boxed{4\ln\left|\frac{x-2}{x+1}\right| + C}$ or $\boxed{2\ln\left|\frac{x-1}{x+3}\right| + C}$.

5. (a) Sketch the graph of the function and shade the region whose area is represented by the integral below. **Label all pertinent information.** Do not evaluate.

$$\int_{-3}^{4} (2x+15) - x^2 \text{ dx}$$

Solution: Answer: `https://www.desmos.com/calculator/vem9dirors`.

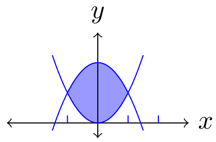
(b) Consider the region bounded by $y = x^2,\ y = 2 - x^2$.

i. Find the intersection points of the two curves.

Solution: Answer: $\boxed{-1 \text{ and } 1}$.

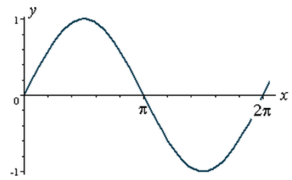
ii. Sketch the two curves and shade the region bounded by the two curves.

Solution:



iii. Set up, but **do not evaluate** an integral for the area of the shaded region.

Solution: Answer: $\boxed{\int_{-1}^{1} 2-2x^2 \text{ dx}}$



(c) For your convenience, the graph of $y = \sin(x)$ is shown.

Set up the definite integral for the area of the region bounded by the curves $y = \sin(x)$, $y = 0$, $x = 0$ and $x = \frac{\pi}{2}$. Then evaluate the area.

Solution: Answer: area = $\boxed{1}$.

6. Write down the first key step/s of evaluating the following the integrals (There often are more than one right answer). You don't need to evaluate the integrals.

(a) $\int_0^4 \frac{\ln(x)}{\sqrt{x}} \text{ dx}$ ____________________

Solution: Answer: Integration by parts with $u = \ln(x)$ and $dv = \frac{1}{\sqrt{x}}$.

(b) $\int \frac{1}{x\ln(x)} \text{ dx}$ ____________________

Solution: Answer: u-substitution for $\ln(x)$.
At the end you should get $\boxed{\ln(\ln(x)) + \text{ Constant}}$.

(c) $\int_1^2 \ln(x) \text{ dx}$ ____________________

Solution: Integration by parts with (the only option) $u = \ln(x)$ and $dv = \text{ dx}$

(d) $\int xe^{0.2x} \text{ dx}$ ____________________

Solution: Integration by parts with $u = x$ and $dv = e^{0.2x} \text{ dx}$

(e) $\int_0^1 e^x \sin(x) \text{ dx}$ ____________________

Solution: Integration by parts with either $\boxed{u = e^x \text{ and } dv = \sin(x)\ \mathrm{dx}}$ OR $\boxed{dv = e^x\ \mathrm{dx} \text{ and } u = \sin(x)}$. Then repeat.

(f) $\int \frac{1}{x^2 + 2x + 4}$ dx ____________________

Solution: Complete the square to get $\int \frac{1}{(x+1)^2 + 3}$ dx, then use inverse trig substitution with $\boxed{x + 1 = \sqrt{3}\,tan(\theta)}$ (fewer negative signs to work with) or $\boxed{x + 1 = \sqrt{3}\cot(\theta)}$

7. (Improper integrals Sec 7.8)
   (a) When is an integral improper? Hint: There are two kinds. Copy definitions from Sec 7.8 pg 527 and 531.
   (b) When is an integral proper? (When it is not improper, but explain what needs to happen for a definite integral to be proper).
   (c) True or False, and why? Let $f(x)$ be continuous everywhere. $\int_a^\infty f(x)\ \mathrm{dx} = \lim_{t\to\infty} \int_a^t f(x)\ \mathrm{dx}$ **T** **F**

   Justification:

   Solution: Answer: $\boxed{\text{True}}$, by definition

   (d) True or False, and why? Let $f(x)$ be continuous everywhere. $\int_a^\infty f(x)\ \mathrm{dx} = \lim_{a\to\infty} \int_a^t f(x)\ \mathrm{dx}$ **T** **F**

   Justification:

   Solution: Answer: $\boxed{\text{False}}$, by definition

   (e) True or False, and why? The integral $\int_2^3 \sqrt{x-2}$ dx is improper. **T** **F**

   Solution: Answer: $\boxed{False}$. The function $\sqrt{x-2}$ is continuous on $[2, 3]$.

   (f) True or False, and why? $\int_0^1 \frac{27}{x^5}$ dx is improper. **T** **F**

   Solution: Answer: $\boxed{True}$. The function $\frac{27}{x^5}$ has an infinite discontinuity at 0 since $\lim_{x\to 0} \frac{27}{x^5}$ does not exist. $\lim_{x\to 0^+} \frac{27}{x^5} = \infty$

   (g) Evaluate $\int_0^1 \frac{27}{x^5}$ dx

   Solution: Answer: $\lim_{a\to 0^+} \int_a^1 \frac{27}{x^5} = \infty$. $\boxed{\text{The integral diverges}}$.

   (h) True or False? $\int_{-1}^1 \frac{1}{x}$ dx is improper. **T** **F**

   Solution: Answer: $\boxed{True}$. The function $\frac{1}{x}$ has an infinite discontinuity at 0 since $\lim_{x\to 0} \frac{1}{x}$ does not exist. $\lim_{x\to 0^+} \frac{1}{x} = \infty$.

   (i) Evaluate $\int_{-1}^1 \frac{1}{x}$ dx

Solution: Answer: $\int_0^1 \frac{1}{x}\,\mathrm{dx} = \lim_{a\to 0^+} \int_a^1 \frac{1}{x} = \infty$. $\boxed{\text{The integral diverges}}$.

(j) Determine whether $\int_0^1 9x^2 \ln(x)\,\mathrm{dx}$ converges or diverges. If it converges, evaluate it.

Solution: Answer: -1

(k) Evaluate $\int_4^8 \frac{4}{x\sqrt{x^2-16}} dx$ OR $\int_{-7}^7 \frac{1}{\sqrt{49-x^2}} dx$. If it converges, evaluate it.

Solution: $\int_4^8 \frac{4}{x\sqrt{x^2-16}} dx$ ans:$\pi/3$, $\int_{-7}^7 \frac{1}{\sqrt{49-x^2}} dx$. ans:$\pi$

(l) Determine whether $\int_e^\infty \frac{1}{x(\ln x)^3}$ dx converges or diverges. If it converges, evaluate the integral.

Solution: Answer: 1/2

(m) Determine whether

$$\int_2^\infty \left(\frac{1}{e^5}\right)^x \mathrm{dx}$$

is convergent or divergent. If it is convergent, evaluate it.

Solution: Answer: $\boxed{1/(5\ 10^e)}$

(n) Determine whether $\int_2^\infty \frac{1}{x^2+8x-9}$ dx is convergent or divergent. If it is convergent, evaluate it.

Solution: Use partial fraction decomposition (probably faster) or complete the square + trig substitution (probably longer). Answer: $\boxed{\ln(11)/10}$.

(o) Determine whether $\int_0^1 \frac{4}{x^5}$ dx is convergent or divergent. If it is convergent, evaluate it.

Solution: Answer: $\boxed{\text{divergent}}$

(p) Determine whether $\int_0^1 \frac{4}{x^{0.5}}$ dx is convergent or divergent. If it is convergent, evaluate it.

Solution: Answer: $\boxed{8}$

(q) Determine whether $\int_2^3 \frac{2}{\sqrt{3-x}}$ dx is convergent or divergent. If it is convergent, evaluate it.

Solution: Answer: $\boxed{4}$

(r) Write 2 improper integrals (different from above) so that one is convergent and the other is divergent.

(s) Write 2 proper (definite) integrals that are different from above.

(t) Write 2 indefinite integrals.

2 Part 3: 11.3 Integral Test and Estimates of Sum

8. Suppose f is a continuous, positive, and decreasing function on $[1, \infty)$. Suppose $a_k = f(k)$ for $k = 1, 2, 3, \ldots$.
   (a) Draw pictures for illustrating the quantities of each of the following. $\int_1^6 f(x)\ \mathrm{dx}$ $\sum_{k=2}^6 a_k$ $\sum_{k=1}^5 a_k$.
   (b) Then rank the three quantities in increasing order.
   (c) What are the conditions needed to apply the Integral Test ?

Solution: From pages 1-2 of `https://www.dropbox.com/s/h4khh34xeyvr126/week7d3_notes11_3part1.pdf?dl=0`

9. Determine whether $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^5}$ converges or diverges.

   a. Explain why the integral test can be applied.

Solution: Let $f(x) = \dfrac{1}{x(\ln x)^5}$. Then $f(x)$ is continuous and positive on for $x \geq 2$. It is also decreasing on $[2, \infty)$ since $x(\ln x)^5$ is a product of increasing functions on $[2, \infty)$. Thus we can use the integral test.

   b. Let $b > 2$. Evaluate $\int_2^b \frac{1}{x(\ln x)^5}$.

Solution: Use u-substitution $u = \ln(x)$.

   c. Evaluate $\displaystyle\int_2^{\infty} \frac{1}{x(\ln x)^5}$.

Solution: Use u-substitution $u = \ln(x)$.

   d. Apply the integral test to determine whether $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^5}$ converges or diverges.

Solution: Since $\displaystyle\int_2^{\infty} \frac{dx}{x(\ln x)^5}$ converges by part (c), we conclude by the integral test that $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^5}$ also converges.

10. Consider the series $\displaystyle\sum_{n=1}^{\infty} \frac{n}{3^n}$.

   a. Verify that the integral test *can* be used to decide if this series converges.

Solution:
Let $f(x) = x/3^x$. We will show that $f(x)$ is positive, continuous, and decreasing on $[1, \infty)$.

This function is positive and continuous for $x \geq 1$ since x is a polynomial, 3^x is an exponential function, and 3^x is never 0 on $[1, \infty)$.

To show that $f(x)$ is decreasing on $[1, \infty)$, compute $f'(x) = \dfrac{1 - x\ln 3}{3^x}$, which implies $f'(x) < 0$ for $x > \dfrac{1}{\ln 3}$. Since $\ln 3 > \ln e = 1$, we have $\dfrac{1}{\ln 3} < 1$, so $f'(x) < 0$ for $x \leq 1$. This justifies the use of the integral test on $\displaystyle\sum_{n=1}^{\infty} \frac{n}{3^n}$.

b. Apply the Integral Test (or another test if you prefer) to prove that this series converges.

Solution: compute $\int \frac{x}{3^x}\,dx$, try integration by parts:
either $\boxed{u = x \text{ and } dv = dx/3^x = 3^{-x}\,dx}$ or $\boxed{u = dx/3^x = 3^{-x} \text{ and } dv = xdx}$.
You should get

$$\int_1^\infty \frac{x}{3^x}\,dx \quad = \quad \lim_{b\to\infty}\left(-\frac{b}{3^b\ln 3} - \frac{1}{3^b(\ln 3)^2} + \frac{1}{3\ln 3} + \frac{1}{3(\ln 3)^2}\right).$$

Since $\lim_{b\to\infty} b/3^b = 0$ by L'Hospital's Rule and $\lim_{n\to\infty} 1/3^b = 0$,

$$\int_1^\infty \frac{x}{3^x}\,dx = \frac{1}{3\ln 3} + \frac{1}{3(\ln 3)^2}.$$

Since $\int_1^\infty \frac{x}{3^x}\,dx$ converges, the series $\sum_{n=1}^{\infty} \frac{n}{3^n}$ also converges by the Integral Test.

c. Determine an explicit upper bound for the remainder R_N when estimating the series by the Nth partial sum. Your answer will depend on N.

Solution: The Nth remainder R_N is at most $\int_N^\infty \frac{x}{3^x}\,dx = \lim_{b\to\infty}\int_N^b \frac{x}{3^x}\,dx = \frac{N\ln 3 + 1}{3^N(\ln 3)^2}$.
See Week 7 day 3 notes
`https://www.dropbox.com/s/phe1v0p04z9q5xv/week7d3_notes11_3part2_plus_hw11_3problem3.pdf?dl=0` (end of file)

d. Find an N for which the upper bound on R_N in part (c) is less than 0.2, and then compute the Nth partial sum s_N to 5 digits after the decimal point.

Solution: See Week 7 day 3 notes
`https://www.dropbox.com/s/phe1v0p04z9q5xv/week7d3_notes11_3part2_plus_hw11_3problem3.pdf?dl=0` (end of file)

11. (Integral Test from 11.3 WebAssign)

(a) Find the values of p for which the integral $\int_e^\infty \frac{6}{x(\ln x)^p}$ dx converges. Evaluate the integral for these values of p.
(Hint: Check what happens when $p = 1$, when $p < 1$, and when $p > 1$.)

Solution: Answer: $\boxed{p > 1 \text{ converges. Otherwise, diverges.}}$

(b) Evaluate the integral $\int_1^\infty \frac{3}{x^6}$ dx. Are the conditions for the Integral Test satisfied? If so, use the Integral Test to determine whether the series $\sum_1^\infty \frac{3}{n^6}$ is convergent or divergent.

Solution: Answer: $= \boxed{3/5}$

(c) Evaluate the integral $\int_1^\infty \frac{1}{(4x+2)^3}$ dx. Are the conditions for the Integral Test satisfied? If so, use the Integral Test to determine whether the series $\sum_1^\infty \frac{1}{(4n+2)^3}$ is convergent or divergent.

Solution: Answer: $= \boxed{1/288}$

(d) Evaluate the integral $\displaystyle\int_1^\infty \frac{1}{\sqrt{x+9}}$ dx. Are the conditions for the Integral Test satisfied? If so, use the Integral Test to determine whether the series $\displaystyle\sum_1^\infty \frac{1}{\sqrt{n+9}}$ is convergent or divergent.

Solution: Answer: $\boxed{\text{divergent}}$

(e) Evaluate the integral

$$\int_1^\infty x\, e^{-9x}\ \mathrm{dx}$$

Are the conditions for the Integral Test satisfied? If so, use the Integral Test to determine whether the series $\displaystyle\sum_1^\infty ne^{-9n}$ is convergent or divergent.

Solution: Answer: $\boxed{10/81e^9}$

(f) The following statement is false: "If $a_n = f(n)$ where $f(x)$ is continuous, positive, and decreasing for $x \geq 1$, and $\displaystyle\int_1^\infty f(x)\,dx$ converges then $\displaystyle\sum_{n=1}^\infty a_n = \int_1^\infty f(x)\,dx$." Give a counterexample by coming up with a continuous, positive, and decreasing $f(x)$ on $[1,\infty)$ and computing both $\displaystyle\sum_{n=1}^\infty a_n$ (where $a_n := f(n)$) and $\int_1^\infty f(x)\,dx$, showing that they are not equal.
(Hint: you know how to compute precisely the sum of any convergent geometric series).

Solution: Counterexample: Let $f(x) = \frac{1}{2^x}$. The sum of the series $\displaystyle\sum_{n=0}^\infty \frac{1}{2^n}$ is $\frac{1}{1-\frac{1}{2}} = 2$ (think the infinite mathematicians joke), so $\displaystyle\sum_{n=1}^\infty \frac{1}{2^n} = 2 - 1 = 1$, while $\displaystyle\int_1^\infty \frac{dx}{2^x} = \lim_{b\to\infty}\left(\frac{1}{2^x(-\ln 2)}\Bigg|_1^b\right) = \frac{1}{2\ln 2} \neq 1.$

3 Part 4: 11.5 Alternating Series and Alt. Ser. Estimation Thm

(Hint: The theorems from Sec 11.5 are given in the fact sheet. See also notes from Week 8 Monday, Sec 11.5 part 2)

12. Consider the series $\displaystyle\sum_{n=0}^\infty \frac{(-1)^n}{n!}$. Recall that the symbol 0! means the number 1.

(a) Determine whether the series $\displaystyle\sum_{n=0}^\infty \frac{(-1)^n}{n!}$ converges or diverges.

Solution: Answer: Sec 11.5, Example 4 on page 735.

(b) Let $b_n = \frac{1}{n!}$. Your computing tool has computed for you $b_7 = \frac{1}{5040}$. What N do you need to use so that the partial sum S_N is correct (to the actual sum of $\displaystyle\sum_{n=0}^\infty \frac{(-1)^n}{n!}$) to three decimal places? Translation: we want $|S_N - S| < 0.0005$.

Solution: Answer: $\boxed{\text{N=6}}$. Alternating Estimation Theorem tells you that $S - S_6 \leq b_7$, and I've computed for you $b_7 = \dfrac{1}{5040}$. Since

$$b_7 = \frac{1}{5040} < \frac{2}{10000} = 0.0002 < 0.0005,$$

we know that we only need to compute the partial sum $S_6 = \sum_{n=0}^{6} \dfrac{(-1)^n}{n!}$. More detailed explanation in Sec 11.5, Example 4 on page 735.

13. For the following questions, circle TRUE or FALSE. Justify briefly.

(a) Suppose $b_k > 0$ for all k and $\sum_{k=1}^{\infty}(-1)^k\, b_k$ is a convergent with sum S and partial sum S_n. Then $|S - S_5| \leq b_6$. **T** **F**

Solution: Answer: True, by the Alternating Series Estimate Theorem

(b) Suppose $b_k > 0$ for all n and $\sum_{k=1}^{\infty}(-1)^k\, b_k$ is a convergent with sum S and partial sum S_n. Then $|S - S_5| \geq b_6$. **T** **F**

Solution: Answer: False. The Alternating Series Estimate Theorem states that the inequality should go the other way.

14. Consider the series $\sum_{n=2}^{\infty}(-1)^n \dfrac{1}{n^3}$ and $\sum_{n=1}^{\infty} \dfrac{(-1)^{n-1}}{2n-1}$. Circle all true statement/s and cross out all false statement/s. (Hint: See the theorems on the exam's fact sheet).

a. The series $\sum_{n=1}^{\infty} \dfrac{(-1)^{n-1}}{2n-1}$ converges.

Solution: $\boxed{\text{True}}$ by the Alternating Series Test.
Thinking about the problem:
The series starts off as $1 - \dfrac{1}{3} + \dfrac{1}{5} - \dfrac{1}{7} + \cdots$ and is alternating with $b_n = \dfrac{1}{2n-1}$. We will check the conditions for the Alternating Series Test.
Doing the problem:
For $b_n = \dfrac{1}{2n-1} > 0$ we need to check $b_{n+1} \leq b_n$ for all n and $b_n \to 0$ as $n \to \infty$.
The inequality $b_{n+1} \leq b_n$ is the same as $\dfrac{1}{2n+1} \leq \dfrac{1}{2n-1}$, which is equivalent to saying $2n + 1 \geq 2n - 1$, and that last inequality is true. Alternatively, using calculus, the function $f(x) = \dfrac{1}{2x-1}$ has derivative $f'(x) = -\dfrac{2}{(2x-1)^2}$, which is negative for $x \geq 1$, so $f(x)$ is decreasing for $x \geq 1$.
For the limit, $\lim\limits_{n\to\infty} b_n = \lim\limits_{n\to\infty} \dfrac{1}{2n-1} = 0$.
We can now use the Alternating Series Test to conclude that $\sum_{n=1}^{\infty} \dfrac{(-1)^{n-1}}{2n-1}$ converges.

b. The series $\sum_{n=2}^{\infty}(-1)^n \dfrac{1}{n^3}$ converges.

Solution: $\boxed{\text{True}}$ by the Alternating Series Test.
Thinking about the problem:
The series is alternating with $b_n = \dfrac{1}{n^3}$. We will check the conditions for the Alternating Series Test.

Doing the problem:

For $b_n = \dfrac{1}{n^3} > 0$ we need to check $b_{n+1} \le b_n$ for all n and $b_n \to 0$ as $n \to \infty$.

The inequality $b_{n+1} \le b_n$ is the same as $\dfrac{1}{(n+1)^3} \le \dfrac{1}{n^3}$, which is equivalent to saying $n+1 \ge n$, and that last inequality is true. Alternatively, using calculus, the function $f(x) = \dfrac{1}{x^3}$ has derivative $f'(x) = -\dfrac{3}{x^4}$, which is negative for $x \ge 2$, so $f(x)$ is decreasing for $x \ge 2$.

For the limit, $\lim\limits_{n\to\infty} b_n = \lim\limits_{n\to\infty} \dfrac{1}{n^3} = 0$.

We can now use the Alternating Series Test to conclude that $\displaystyle\sum_{n=2}^{\infty} \frac{(-1)^n}{n^3}$ converges.

c. Suppose $S = \displaystyle\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{2n-1}$ and $S_{1000} := \displaystyle\sum_{n=1}^{1000} \frac{(-1)^{n-1}}{2n-1}$, $S_{1001} := \displaystyle\sum_{n=1}^{1001} \frac{(-1)^{n-1}}{2n-1}$ are partial sums, as usual. Then is the following True or False, and why?

$$S_{1000} < S < S_{1001}.$$

Solution: $\boxed{\text{True}}$. Explanation: Every alternating series whose terms in absolute value satisfy $b_{n+1} < b_n$ lies in between consecutive partial sums. See Sec 11.5, Figs. 1 and 2, pg 733-734. Thus S is between S_{1000} and S_{1001}. The first term of $\displaystyle\sum_{n=1}^{\infty} (-1)^{n-1} \frac{(-1)^{n-1}}{2n-1}$ is positive, so $S_{1000} < S < S_{1001}$.

d. Suppose $S = \displaystyle\sum_{n=2}^{\infty} (-1)^n \frac{1}{n^3}$ and $S_{1000} := \displaystyle\sum_{n=2}^{1000} (-1)^n \frac{1}{n^3}$, $S_{1001} := \displaystyle\sum_{n=2}^{1001} (-1)^n \frac{1}{n^3}$ are partial sums, as usual. Then is the following True or False, and why?

$$S_{1000} < S < S_{1001}.$$

Solution: $\boxed{\text{True}}$. Explanation: Every alternating series whose terms in absolute value satisfy $b_{n+1} < b_n$ lies in between consecutive partial sums. See Sec 11.5, Figs. 1 and 2, pg 733-734. Thus S is between S_{1000} and S_{1001}. The first term of $\displaystyle\sum_{n=2}^{\infty} (-1)^n \frac{1}{n^3}$ is positive, so $S_{1000} < S < S_{1001}$.