Contents

1. (Limit laws and L'Hospital's Rule Sec 4.4)

   i.) If $\lim_{x\to\infty} f(x) = 0$, then $\lim_{x\to\infty} (xf(x))$ is ...
   (a) zero (b) ∞ (c) non-zero constant (d) another method is needed to determine this

   ii.) If $\lim_{x\to\infty} f(x) = \infty$, then $\lim_{x\to\infty} (f(x) - x)$ is ...
   (a) zero (b) ∞ (c) non-zero constant (d) another method is needed to determine this

   iii.) If $\lim_{x\to 0} f(x) = 0$, then $\lim_{x\to 0} (f(x) - x)$ is ...
   (a) zero (b) ∞ (c) non-zero constant (d) another method is needed to determine this

   iv.) If $\lim_{x\to 0} f(x) = \infty$ and $\lim_{x\to 0} g(x) = \infty$, then $\lim_{x\to 0} (f(x) + g(x))$ is ...
   (a) zero (b) ∞ (c) non-zero constant (d) another method is needed to determine this

   v.) If $\lim_{x\to\infty} f(x) = 1$, then $\lim_{x\to\infty} (f(x))^x$ is ...
   (a) zero (b) ∞ (c) 1 (d) another method is needed to determine this.

   vi.) Evaluate $\lim_{x\to 1} \frac{\ln x}{x-1}$ and $\lim_{x\to\infty} \frac{e^x}{x^2}$.

   vii.) Evaluate $\lim_{x\to\infty} \frac{\ln x}{\sqrt{x}}$ and $\lim_{x\to 0} \frac{\tan x - x}{x^3}$.

   viii.) Evaluate $\lim_{x\to\pi^-} \frac{\sin x}{1-\cos x}$.

   ix.) Evaluate $\lim_{x\to 0^+} x \ln x$. Note: This shows up frequently when we compute our improper integral examples.

   x.) Evaluate $\lim_{x\to 0^+} x(\ln x)^3$. Note: This shows up frequently when we compute our improper integral examples.

1 Part 2: 7.1-7.4 Integration Techniques, 7.8 Improper Integrals

2. (Various definite and indefinite integrals).

   (a) Is $\int_1^2 t^3 \ln(t)dt$ a proper or improper integral?

   (b) Evaluate $\int t^3 \ln(t)dt$ and $\int_1^2 t^3 \ln(t)dt$

   (c) Perform either a proof or a reality check for the previous problem. For example, differentiate your answer (for a proof) or check that your definite integral is a positive number, since $t^3 ln(t)$ is positive for $t > 1$.

   (d) $\int \frac{2x-3}{8+x^2}$ dx

   (e) $\int \frac{\sin(\ln(x))}{x}$ dx

   (f) Is $\int_0^1 x\ e^{-x^2}$ a proper or improper integral?

   (g) Evaluate $\int_0^1 x\ e^{-x^2}$

   (h) Perform a reality check for your answer to the previous problem.

   (i) Evaluate $\int (x+2)\sin(3x)$ dx

   (j) Prove that your previous answer is correct by differentiating and applying the Fundamental Theorem of Calculus.

(k) Evaluate $\int e^{\sqrt{x}}$ dx

(l) Prove that your previous answer is correct by differentiating and applying the Fundamental Theorem of Calculus.

(m) Evaluate $\int \frac{x}{\sqrt{4-x^2}}$

(n) Prove that your previous answer is correct by differentiating and applying the Fundamental Theorem of Calculus.

(o) Is $\int_0^9 \frac{1}{\sqrt{x}}$ dx an improper integral?

(p) Determine whether $\int_0^{25} \frac{1}{\sqrt{x}}$ dx or $\int_0^{16} \frac{1}{\sqrt{x}}$ dx or $\int_0^{9} \frac{1}{\sqrt{x}}$ dx is convergent or divergent. If it is convergent, evaluate the integral.

(q) True of false? Let $a > 0$. $\int_0^a f(x) = \int_0^a f(a-x)$ dx. **T** **F**
If T, justify. If F, give a counterexample.

(r) True of false? Let $a > 0$. $\int_0^a f(x) = \int_0^a f(x-a)$ dx. **T** **F**
If T, justify. If F, give a counterexample.

(s) Write a sanity-check-type calculation (different from what you've written above) to further confirm your answer in the previous two questions.

3. (From class handouts)

(a) Is the integral $\int_0^\pi \sin^3(5x)\,dx$ a proper or improper integral?

(b) Evaluate $\int_0^\pi \sin^3(5x)\,dx$.

(c) Evaluate $\int \frac{dx}{\sqrt{9-x^2}}$.

(d) Consider $\int_0^{\sqrt{2}} \frac{x^2}{\sqrt{4-x^2}}dx$, $\int_{-\sqrt{2}}^{0} \frac{x^2}{\sqrt{4-x^2}}dx$, $\int_0^{2} \frac{x^2}{\sqrt{4-x^2}}dx$ and $\int_{-2}^{\sqrt{2}} \frac{x^2}{\sqrt{4-x^2}}dx$. Which are proper integrals and which are improper integrals?

(e) Evaluate $\int_0^{\sqrt{2}} \frac{x^2}{\sqrt{4-x^2}}dx$.

(f) Without explicitly trying to evaluate this integral, determine whether it is possible for $\int_0^\infty x^2e^{-x}\,dx$ to be convergent to a negative value.

(g) Is $\int_0^\infty x^2e^{-x}\,dx$ convergent or divergent? If convergent, evaluate it.

(h) (Note: You may use the fact sheets to look up derivatives and integrals of trig functions)
Evaluate 1.) $\int_0^{\frac{\pi}{2}} \sin^5 x$ dx or 2.) $\int \frac{\cos^5 x}{\sin x}$ dx or 3.) $\int_0^\pi \cos^4(2x)$ dx or
4.) $\int \sin^3 x \cos^5 x$ dx or 5.) $\int \sin^2 x \cos^2 x$ dx or 6.) $\int \tan^3 x \sec^3 x$ dx or
7.) $\int \tan^2 x \sec^4 x$ dx or 8.) $\int_0^{\frac{\pi}{4}} \sec^4 x$ dx or 9.) omitted or 10.) $\int \tan^3 x \sec^4 x$ dx.

(i) Evaluate 1.) $\int \frac{1}{x\sqrt{4-x^2}}$ dx or 2.) $\int \frac{1}{\sqrt{x^2+16}}$ dx or 3.) $\int_{\sqrt{2}}^2 \left(\frac{1}{x^3\sqrt{x^2-1}}\right)$ dx or
4.) $\int_0^{\frac{\pi}{2}} \frac{\cos x}{\sqrt{1+\sin^2 x}}$ dx or 5.) $\int_0^1 \left(\frac{1}{\sqrt{-x^2+2x+3}}\right)$ dx (Hint: First complete the square) or
6.) $\int \frac{1}{\sqrt{1+16x^2}}$ dx or 7.) $\int_0^{\sqrt{2}} \frac{x^2}{\sqrt{4-x^2}}$ dx or 8.) $\int \frac{1}{x^2\sqrt{9x^2-1}}$ dx or
9.) $\int \sqrt{5+4x-x^2}$ dx. (Hint: First complete the square)

(j) 1.) Evaluate $\int \ln\left(x+\sqrt{1+x^2}\right)$ dx or 2.) $\int x \tan^2 x$ dx or 3.) $\int \cos(\sqrt{x})$ dx or
4.) Evaluate $\int x^2 (\ln x)^2$ dx or 5.) omitted or 6.) $\int \cos(\ln x)$ dx or
7a.) How can you derive the formula for Integration by Parts? or
7b.) Evaluate $\int_0^{\frac{\pi}{2}} x \cos(2x)$ dx or
7c.) Suppose $f(1)=2$, $f(4)=7$, $f'(1)=5$, $f'(4)=3$. Suppose f'' is continuous. Evaluate $\int_1^4 x f''(x)$ dx
or 7d.) Evaluate $\int \arctan x$ dx or 7e.) $\int e^x \cos x$ dx or
7f.) A particle that moves along a straight line has velocity $v(t) = t^3 e^{-t}$ meters per second after t seconds. How far will it travel during the first t seconds?

(k) Evaluate 1.) $\int_0^\infty e^{-2x}$ dx or 2.) $\int_1^\infty \frac{1}{\sqrt{x}}$ or 3.) $\int_1^\infty \sin^2 x$ dx or 4.) $\int_1^\infty \frac{1}{x^2+2x-3}$ dx.

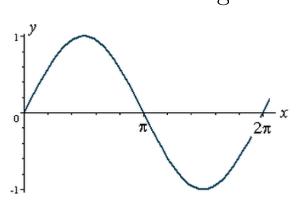
(l) 1.) Evaluate $\int_0^8 \frac{1}{\sqrt[3]{x}}$ dx or 2.) Evaluate $\int_1^{\frac{\pi}{2}} \sec x$ or 3.) Evaluate $\int_0^5 \frac{x}{x-2}$ dx or
4.) Use the Comparison Theorem to determine whether $\int_1^\infty \frac{x}{x^3+1}$ dx.

4. (Integrating rational functions)
   (a) Evaluate $\int \frac{2x+1}{x^2-4}\,dx$.
   (b) Decompose $\frac{3x^2+2x-3}{x^3-x}$ into partial fractions.
   (c) Provide a computation that is either a formal verification (that is, a proof) or simply a reality-check for your answer to the previous question.
   (d) Evaluate $\int \frac{10}{(x+5)(x-2)}$ dx.
   (e) Provide a computation that is either a formal verification (that is, a proof) or simply a sanity-check for your answer to the previous question.
   (f) Evaluate $\int \frac{9}{(x-6)(x+3)}$ dx or $\int \frac{12}{(x-2)(x+1)}$ dx or $\int \frac{8}{(x-1)(x+3)}$ dx.

5. (a) Sketch the graph of the function and shade the region whose area is represented by the integral below. **Label all pertinent information.** Do not evaluate.

$$\int_{-3}^{4} (2x+15) - x^2 \text{ dx}$$

   (b) Consider the region bounded by $y = x^2$, $y = 2 - x^2$.
      i. Find the intersection points of the two curves.
      ii. Sketch the two curves and shade the region bounded by the two curves.
      iii. Set up, but **do not evaluate** an integral for the area of the shaded region.



   (c) For your convenience, the graph of $y = \sin(x)$ is shown.
   Set up the definite integral for the area of the region bounded by the curves $y = \sin(x)$, $y = 0$, $x = 0$ and $x = \frac{\pi}{2}$. Then evaluate the area.

6. Write down the first key step/s of evaluating the following the integrals (There often are more than one right answer). You don't need to evaluate the integrals.

(a) $\int_0^4 \frac{\ln(x)}{\sqrt{x}}$ dx ______________________

(b) $\int \frac{1}{x\ln(x)}$ dx ______________________

(c) $\int_1^2 \ln(x)$ dx ______________________

(d) $\int xe^{0.2x}$ dx ______________________

(e) $\int_0^1 e^x \sin(x)$ dx ______________________

(f) $\int \frac{1}{x^2+2x+4}$ dx ______________________

7. (Improper integrals Sec 7.8)

(a) When is an integral improper? Hint: There are two kinds. Copy definitions from Sec 7.8 pg 527 and 531.

(b) When is an integral proper? (When it is not improper, but explain what needs to happen for a definite integral to be proper).

(c) True or False, and why? Let $f(x)$ be continuous everywhere. $\int_a^\infty f(x)$ dx $= \lim_{t\to\infty} \int_a^t f(x)$ dx **T** **F**

Justification:

(d) True or False, and why? Let $f(x)$ be continuous everywhere. $\int_a^\infty f(x)$ dx $= \lim_{a\to\infty} \int_a^t f(x)$ dx **T** **F**

Justification:

(e) True or False, and why? The integral $\int_2^3 \sqrt{x-2}$ dx is improper. **T** **F**

(f) True or False, and why? $\int_0^1 \frac{27}{x^5}$ dx is improper. **T** **F**

(g) Evaluate $\int_0^1 \frac{27}{x^5}$ dx

(h) True or False? $\int_{-1}^1 \frac{1}{x}$ dx is improper. **T** **F**

(i) Evaluate $\int_{-1}^1 \frac{1}{x}$ dx

(j) Determine whether $\int_0^1 9x^2 \ln(x)$ dx converges or diverges. If it converges, evaluate it.

(k) Evaluate $\int_4^8 \frac{4}{x\sqrt{x^2-16}}dx$ OR $\int_{-7}^7 \frac{1}{\sqrt{49-x^2}}dx$. If it converges, evaluate it.

(l) Determine whether $\int_e^\infty \frac{1}{x(\ln x)^3}$ dx converges or diverges. If it converges, evaluate the integral.

(m) Determine whether

$$\int_2^\infty \left(\frac{1}{e^5}\right)^x \text{ dx}$$

is convergent or divergent. If it is convergent, evaluate it.

(n) Determine whether $\int_2^\infty \frac{1}{x^2+8x-9}$ dx is convergent or divergent. If it is convergent, evaluate it.

(o) Determine whether $\int_0^1 \frac{4}{x^5}$ dx is convergent or divergent. If it is convergent, evaluate it.

(p) Determine whether $\int_0^1 \frac{4}{x^{0.5}}$ dx is convergent or divergent. If it is convergent, evaluate it.

(q) Determine whether $\int_2^3 \frac{2}{\sqrt{3-x}}$ dx is convergent or divergent. If it is convergent, evaluate it.

(r) Write 2 improper integrals (different from above) so that one is convergent and the other is divergent.

(s) Write 2 proper (definite) integrals that are different from above.

(t) Write 2 indefinite integrals.

2 Part 3: 11.3 Integral Test and Estimates of Sum

8. Suppose f is a continuous, positive, and decreasing function on $[1, \infty)$. Suppose $a_k = f(k)$ for $k = 1, 2, 3, \ldots$.
   (a) Draw pictures for illustrating the quantities of each of the following. $\int_1^6 f(x)$ dx $\quad \sum_{k=2}^{6} a_k \quad \sum_{k=1}^{5} a_k$.
   (b) Then rank the three quantities in increasing order.
   (c) What are the conditions needed to apply the Integral Test ?

9. Determine whether $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^5}$ converges or diverges.
   a. Explain why the integral test can be applied.
   b. Let $b > 2$. Evaluate $\int_2^b \frac{1}{x(\ln x)^5}$.
   c. Evaluate $\displaystyle\int_2^{\infty} \frac{1}{x(\ln x)^5}$.
   d. Apply the integral test to determine whether $\displaystyle\sum_{n=2}^{\infty} \frac{1}{n(\ln n)^5}$ converges or diverges.

10. Consider the series $\displaystyle\sum_{n=1}^{\infty} \frac{n}{3^n}$.
    a. Verify that the integral test *can* be used to decide if this series converges.
    b. Apply the Integral Test (or another test if you prefer) to prove that this series converges.
    c. Determine an explicit upper bound for the remainder R_N when estimating the series by the Nth partial sum. Your answer will depend on N.
    d. Find an N for which the upper bound on R_N in part (c) is less than 0.2, and then compute the Nth partial sum s_N to 5 digits after the decimal point.

11. (Integral Test from 11.3 WebAssign)
    (a) Find the values of p for which the integral $\displaystyle\int_e^{\infty} \frac{6}{x(\ln x)^p}$ dx converges. Evaluate the integral for these values of p.
    (Hint: Check what happens when $p = 1$, when $p < 1$, and when $p > 1$.)
    (b) Evaluate the integral $\displaystyle\int_1^{\infty} \frac{3}{x^6}$ dx. Are the conditions for the Integral Test satisfied? If so, use the Integral Test to determine whether the series $\displaystyle\sum_1^{\infty} \frac{3}{n^6}$ is convergent or divergent.
    (c) Evaluate the integral $\displaystyle\int_1^{\infty} \frac{1}{(4x+2)^3}$ dx. Are the conditions for the Integral Test satisfied? If so, use the Integral Test to determine whether the series $\displaystyle\sum_1^{\infty} \frac{1}{(4n+2)^3}$ is convergent or divergent.
    (d) Evaluate the integral $\displaystyle\int_1^{\infty} \frac{1}{\sqrt{x+9}}$ dx. Are the conditions for the Integral Test satisfied? If so, use the Integral Test to determine whether the series $\displaystyle\sum_1^{\infty} \frac{1}{\sqrt{n+9}}$ is convergent or divergent.

(e) Evaluate the integral

$$\int_1^\infty x\, e^{-9x}\ \mathrm{dx}$$

Are the conditions for the Integral Test satisfied? If so, use the Integral Test to determine whether the series $\sum_1^\infty ne^{-9n}$ is convergent or divergent.

(f) The following statement is false: "If $a_n = f(n)$ where $f(x)$ is continuous, positive, and decreasing for $x \geq 1$, and $\int_1^\infty f(x)\,dx$ converges then $\sum_{n=1}^\infty a_n = \int_1^\infty f(x)\,dx$." Give a counterexample by coming up with a continuous, positive, and decreasing $f(x)$ on $[1, \infty)$ and computing both $\sum_{n=1}^\infty a_n$ (where $a_n := f(n)$) and $\int_1^\infty f(x)\,dx$, showing that they are not equal.
(Hint: you know how to compute precisely the sum of any convergent geometric series).

3 Part 4: 11.5 Alternating Series and Alt. Ser. Estimation Thm

(Hint: The theorems from Sec 11.5 are given in the fact sheet. See also notes from Week 8 Monday, Sec 11.5 part 2)

12. Consider the series $\sum_{n=0}^\infty \frac{(-1)^n}{n!}$. Recall that the symbol 0! means the number 1.

(a) Determine whether the series $\sum_{n=0}^\infty \frac{(-1)^n}{n!}$ converges or diverges.

(b) Let $b_n = \frac{1}{n!}$. Your computing tool has computed for you $b_7 = \frac{1}{5040}$. What N do you need to use so that the partial sum S_N is correct (to the actual sum of $\sum_{n=0}^\infty \frac{(-1)^n}{n!}$) to three decimal places? Translation: we want $|S_N - S| < 0.0005$.

13. For the following questions, circle TRUE or FALSE. Justify briefly.

(a) Suppose $b_k > 0$ for all k and $\sum_{k=1}^\infty (-1)^k\ b_k$ is a convergent with sum S and partial sum S_n. Then $|S - S_5| \leq b_6$. **T** **F**

(b) Suppose $b_k > 0$ for all n and $\sum_{k=1}^\infty (-1)^k\ b_k$ is a convergent with sum S and partial sum S_n. Then $|S - S_5| \geq b_6$. **T** **F**

14. Consider the series $\sum_{n=2}^\infty (-1)^n \frac{1}{n^3}$ and $\sum_{n=1}^\infty \frac{(-1)^{n-1}}{2n-1}$. Circle all true statement/s and cross out all false statement/s. (Hint: See the theorems on the exam's fact sheet).

a. The series $\sum_{n=1}^\infty \frac{(-1)^{n-1}}{2n-1}$ converges.

b. The series $\sum_{n=2}^\infty (-1)^n \frac{1}{n^3}$ converges.

c. Suppose $S = \sum_{n=1}^\infty \frac{(-1)^{n-1}}{2n-1}$ and $S_{1000} := \sum_{n=1}^{1000} \frac{(-1)^{n-1}}{2n-1}$, $S_{1001} := \sum_{n=1}^{1001} \frac{(-1)^{n-1}}{2n-1}$ are partial sums, as usual. Then is the following True or False, and why?

$$S_{1000} < S < S_{1001}.$$

d. Suppose $S = \sum_{n=2}^\infty (-1)^n \frac{1}{n^3}$ and $S_{1000} := \sum_{n=2}^{1000} (-1)^n \frac{1}{n^3}$, $S_{1001} := \sum_{n=2}^{1001} (-1)^n \frac{1}{n^3}$ are partial sums, as usual. Then is the following True or False, and why?

$$S_{1000} < S < S_{1001}.$$