Sections 11.1-11.2, 11.4-11.6

1. For the following questions, circle TRUE or FALSE, and give a justification. True statements should be argued for using facts, theorems or definitions from class.

(a) If $\lim_{n\to\infty} a_n = 0$ then the series $\sum a_n$ converges. **T** **F**

Justification:

Solution: : False. Counterexample: $\frac{1}{\sqrt{n}} \to 0$ as $n \to \infty$, but $\sum \frac{1}{\sqrt{n}}$ diverges.

(b) If $a_n > 0$, $b_n > 0$ for all n, $\sum b_n$ diverges, and $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$, then $\sum a_n$ diverges. **T** **F**

Justification:

Solution: : False. Counterexample: Let $b_n = \frac{1}{n}$ (and so $\sum b_n$ by p-series/harmonic series test) and $a_n = \frac{1}{n^2}$. We have $\lim_{n\to\infty} \frac{a_n}{b_n} = \lim_{n\to\infty} \frac{n}{n^2} = 0$. But $\sum a_n$ converges.

(c) If $a_n > 0$ for all n & $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = 0$, then $\sum a_n$ is convergent by the ratio test **T** **F**

Justification:

Solution: : True because $0 < 1$.

(d) If a_n and b_n are both positive for all n and $\lim_{n\to\infty} \frac{a_n}{b_n} = 0$, then $\sum a_n$ is convergent by the limit comparison test **T** **F**

Justification:

Solution: : False. You cannot conclude this in general (for example, when you don't know that $\sum b_n$ converges). Counterexample: $a_n = n$ and $b_n = n^2$.

(e) The harmonic series $\sum 1/n$ is convergent by the p-series test **T** **F**

Justification:

Solution: : False. The harmonic series is divergent.

(f) We can use the ratio test *alone* to show the geometric series $\sum \frac{2^n}{3^n}$ converges **T** **F**

Justification:

Solution: : True. The ratio of $\frac{2^{n+1}}{3^{n+1}} \frac{3^n}{2^n}$ goes to $\frac{2}{3} < 1$ as $n \to \infty$.

(g) We can use the p-series test *alone* to show the series $\sum 2^n/3^n$ converges **T** **F**

Justification:

Solution: : False because $\sum 2^n/3^n$ does not look like a p-series.

(h) We can apply the monotonic sequence theorem to show that the geometric **sequence** $\{2^n/3^n\}_{n=1}^{\infty}$ is convergent **T** **F**

Justification:

Solution: : True. The sequence $\left\{\left(\frac{2}{3}\right)^n\right\}_{n=1}^{\infty}$ is bounded below (for example, by 0 and -2) and bounded above (for example, by $\frac{2}{3}$ and 5). This sequence is also decreasing. By the monotonic sequence theorem, the sequence is convergent.

(i) We can apply the monotonic sequence theorem to show that the harmonic sequence $\left\{\frac{1}{n}\right\}_{n=1}^{\infty}$ is convergent **T** **F**
Justification:

Solution: : True. The sequence $\left\{\frac{1}{n}\right\}_{n=1}^{\infty}$ is bounded below (for example, by 0 and -2) and bounded above (for example, by 1 and 5). This sequence is also decreasing. By the monotonic sequence theorem, the sequence is convergent.

(j) We can apply the squeeze theorem to show that the alternating harmonic sequence $\left\{\frac{(-1)^n}{n}\right\}$ is convergent **T** **F**
Justification:

Solution: : True. Squeeze each term between $\left\{-\frac{1}{n}\right\}$ and $\left\{\frac{1}{n}\right\}$.

(k) It is impossible for a subset of a line to have infinitely many points and have length zero. **T** **F**
Justification:

Solution: : F. It is possible, for example, the Cantor set, see `http://egunawan.github.io/spring18/hw/problemset_a_s18.pdf`

(l) The divergence of the p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ for $0 < p < 1$ follows from divergence of the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ and the comparison test. **T** **F**
Justification:

Solution: : T. Let $a_n = \frac{1}{n^p}$ (for some $0 < p < 1$) and $b_n = \frac{1}{n^1}$. Then $a_n > b_n$ for all $n = 2, 3, \ldots$. Since $\sum b_n$ diverges (by divergence of the harmonic series), by the comparison test $\sum a_n$ also diverges.

(m) The convergence of the p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ for $p > 1$ follows from divergence of the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ and the comparison test. **T** **F**
Justification:

Solution: F. You can't conclude the convergence of a series by comparing it with a divergent series.

(n) Convergence of $\sum \frac{1}{n^p}$ for $p > 1$ can be shown with the ratio test. **T** **F**
Justification:

Solution: False. Since $\frac{a_{n+1}}{a_n} = \frac{n^p}{(n+1)^p} \to 1$ as $n \to \infty$, the ratio test is inconclusive for this series.

(o) Divergence of a p-series for $p < 1$ can be shown with the ratio test. **T** **F**
Justification:

Solution: False. Since $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{n^p}{(n+1)^p} = 1$, the ratio test is inconclusive for this series.

2. (a) State the contrapositive of the factual statement: "If the sequence $\{a_n\}$ is unbounded, then it is divergent".

Solution: If the sequence $\{a_n\}$ is convergent, then it is bounded.

(b) Is the contrapositive statement you wrote as your answer to part (a) true or false? **Justification (explain or give a counterexample):**

Solution: True. Since the statement in part (a) is true (even though I haven't proven it), the contrapositive statement is also true (because the contrapositive statement is equivalent to the original statement).

(c) The converse of part (a) is the following: "If the sequence $\{a_n\}$ is divergent, then $\{a_n\}$ is unbounded". Is this true or false? **Justification (explain or give a counterexample):**

Solution: False. Counterexample: let $a_n = (-1)^n$.

3. Answer the following on the line provided.

(a) What is the 100th term of the sequence $\{2, 5, 8, 11, \dots\}$?
(The terms 2 and 5 are the first and second term, respectively) ____________

Solution: $\boxed{-1 + 300 = 299}$

(b) Find a formula for the general term a_n of the sequence $\left\{1, -\frac{2}{5}, \frac{3}{25}, -\frac{4}{125}, \frac{5}{625} \cdots\right\}$. Make sure to specify your starting value of n.

Solution: Observe that for $n \geq 1$,

$$a_n = \frac{n}{(-5)^{n-1}}$$

(c) Write the geometric series $4 + 2 + 1 + \frac{1}{2} \cdots$ in standard form.
(summation notation $\sum$) ____________

Solution:

$$\sum_{n=-2}^{\infty} \left(\frac{1}{2}\right)^n \quad \text{or} \quad \sum_{n=-1}^{\infty} \left(\frac{1}{2}\right)^{n-1} \quad \text{or} \quad \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^{n-2}$$

(d) Find the 4th term a_4 in the recursive sequence $a_{n+1} = 2a_n + a_{n-1}$
when $a_1 = 1$ and $a_2 = 1$. ____________

Solution: $\boxed{a_4 = 7}$ because $a_3 = 2a_2 + a_1 = 2\,.\,1 + 1 = 3$, so $a_4 = 2a_3 + a_2 = 2\,.3 + 1 = 7$.

(e) Find the 7th term a_7 in the recursive sequence $a_{n+1} = a_n + a_{n-1}$
when $a_1 = 2$ and $a_2 = 3$. ____________

Solution: $\boxed{a_7 = 34}$

(f) We can use geometric series to compute

$$\frac{1}{10}+\frac{1}{100}+\frac{1}{1000}+\ldots.$$

What fraction is this equal to?

Solution: $\boxed{\frac{1}{9}.}$

(g) We can use geometric series to compute $0.9999\ldots$. What fraction is this equal to?

Solution: $\boxed{\frac{1}{1}.}$

(h) One of the two decimal expansions for a number is $2.449999\ldots$. What's the other?

Solution: $\boxed{2.45}$ See `https://egunawan.github.io/spring18/hw/problemset_a_s18.pdf`

(i) Use geometric series to compute the fraction for $1.833333\ldots$.

Solution: $\boxed{11/6}$

Perform a sanity check against your answer.

Solution: For example, you can check that your answer is bigger than 1.5 but smaller than 2.

(j) Use geometric series to compute the fraction for $1.08333333333333333\ldots$

Solution: $\boxed{13/12}$

Perform a sanity check against your answer.

Solution: For example, you can check that your answer is bigger than 1 but smaller than 1.1.

(k) Does the series $\displaystyle\sum_{n=1}^{\infty} -\ln\left(\frac{n}{2n+7}\right)$ converge or diverge? ____________

Solution: The series <u>diverges</u>. You can apply divergence test because $\displaystyle\lim_{n\to\infty} -\ln\left(\frac{n}{2n+7}\right) = -\ln\left(\frac{1}{2}\right) \neq 0.$

(l) Find the sum of the series $\displaystyle\sum_{k=0}^{\infty} 5\left(\frac{2}{3}\right)^{k}$. ____________

Solution: $\boxed{15}$

(m) Find the sum of the series $\displaystyle\sum_{n=2}^{\infty} 5\left(\frac{(-6)^{n-1}}{7^n}\right)$. ____________

Solution: $\boxed{-\frac{30}{91}.}$

Perform a sanity check against your answer.

Solution: For example, since the first term of the series is a negative number, you can check that your answer is negative.

(n) Find the sum of the series ____________

$$-5+3-\frac{9}{5}+\frac{27}{25}-\frac{81}{125}+\ldots.$$

Solution: *If* you want your first term to be ar^0 (and, consequently, your second term to be ar^1), then you would write $r=-\frac{3}{5}$ with $a=-5$. Therefore the sum of the infinite series is

$$\boxed{-5\frac{1}{1-r}=-5\frac{1}{1+\frac{3}{5}}=-\frac{25}{8}}.$$

Perform a sanity check against your answer.

Solution: For example, since the first term of the series is -5, you can check that your answer is negative.

(o) Write an expression for the nth term in the sequence ____________
$\left\{\frac{1}{2},\frac{1}{6},\frac{1}{24},\frac{1}{120},\ldots\right\}$. (The terms $\frac{1}{2}$ and $\frac{1}{6}$ are the first and second terms in the sequence)

Solution: $\boxed{a_n=\frac{1}{(n+1)!} \text{ for } n=1,2,3,\ldots}$

(p) Write an equivalent series with index summation beginning at $n=0$. ____________
$\sum_{n=2}^{\infty}\frac{2^n}{(n-2)!}$

Solution: $\boxed{\sum_{n=0}^{\infty}\frac{2^{n+2}}{(n)!}}$

Perform a sanity check against your answer.

Solution: For example, you can write down the first two terms of the series and confirm that they match.

(q) Write an equivalent series with index summation beginning at $n=1$. ____________
$\sum_{n=2}^{\infty}\frac{2^n}{(n-2)!}$

Solution: $\boxed{\sum_{n=1}^{\infty}\frac{2^{n+1}}{(n-1)!}}$

Perform a sanity check against your answer.

Solution: For example, you can write down the first two terms of the series and confirm that they match.

(r) For what values of k does the series $\sum \frac{5}{n^k}$ converge? Please explain. ____________

Solution: $\boxed{k > 1, \text{ by the } p\text{-series test.}}$

(s) Find the values of A so that the series $\sum_{n=1}^{\infty} \frac{(A)^{n-1}}{3^{n-1}}$ is convergent. Please explain.

Solution: By the geometric series theorem/test, we need $\left|\frac{A}{3}\right| < 1$, so $\boxed{-3 < A < 3}$.

(t) Find the values of B so that the series $\sum_{n=1}^{\infty} \frac{(B-3)^{n-1}}{3^{n-1}}$ is convergent. Please explain.

Solution: By the geometric series theorem/test, we need $\left|\frac{B-3}{3}\right| < 1$, so $\boxed{0 < B < 6}$.

(u) Find the values of C so that the series $\sum_{n=5}^{\infty} \frac{(C-2)^{n}}{3^{n+1}}$ is convergent. Please explain.

Solution: By the geometric series theorem/test, we need $\left|\frac{C-2}{3}\right| < 1$, so $\boxed{-1 < C < 5}$.

4. The following questions ask you to determine the converge/divergence of a series. To receive credit, give detailed explanation (for example, follow my answer key - to be posted).

(a) Determine whether the series $\sum_{k=1}^{\infty} \frac{k^k}{7^k (k)!}$ is convergent.

Solution: This series converges. You know ratio test will work because you see at least ONE of factorial, exponent, and n^n. See last page of class notes `https://egunawan.github.io/spring18/notes/notes11_6part3.pdf`

(b) Determine whether the series $\sum_{k=1}^{\infty} \frac{k^k}{2^k (k)!}$ is convergent.

Solution: In contrast to (the *very similar* series) above, this series diverges. You know ratio test will work because you see at least ONE of factorial, exponent, and n^n. See last pg of notes `https://egunawan.github.io/spring18/notes/notes11_6part3.pdf`

(c) Determine whether the series $\sum_{n=3}^{\infty} \frac{6}{n\sqrt{n^2-8}}$ converges or diverges.

Solution: Let $a_n = \frac{6}{n\sqrt{n^2-8}}$. Let $b_n = \frac{1}{n^2}$.
Then $\frac{a_n}{b_n} \to 6$ (which is a positive number) as $n \to \infty$.
Since $\sum b_n$ converges by the p-series test, we conclude that $\sum a_n$ converges by limit comparison test.

(d) Determine whether the series $\sum_{n=1}^{\infty} \frac{\sqrt{n^3+1}}{3n^3-4n+2}$ converges or diverges. Make sure to state which test/s you use and provide justifying computations and arguments.

Solution: Let $b_n = \dfrac{1}{n^{3/2}}$. Note that $\sum b_n$ converges since it's a p-series with $p = 3/2 > 1$. Moreover,

$$\begin{aligned}\lim_{n\to\infty} \frac{a_n}{b_n} &= \lim_{n\to\infty} \frac{\sqrt{n^3+1} \quad n^{3/2}}{3n^3 - 4n + 2} \\ &= \lim_{n\to\infty} \frac{\sqrt{n^3\ (n^3+1)}}{3n^3 - 4n + 2} \\ &= \lim_{n\to\infty} \frac{\sqrt{n^6+n^3}}{3n^3 - 4n + 2} \\ &= \frac{1}{3} > 0.\end{aligned}$$

Therefore, since $\sum b_n$ converges, $\sum a_n$ also converges by the Limit Comparison Test.

(e) Determine whether the series $\sum_{n=1}^{\infty} \dfrac{n!}{7^n(n+8)!}$ converges or diverges. Make sure to state which test/s you use and provide justifying computations and arguments.

Solution: The series converges. Both the ratio test and limit comparison test work. Make sure you give detailed explanation.

(f) Determine whether the series $\sum_{n=1}^{\infty} \dfrac{(n+8)!}{7^n n!}$ converges or diverges. Make sure to state which test/s you use and provide justifying computations and arguments. (This looks similar to above, but this isn't a typo).

Solution: The series converges. Both the ratio test and limit comparison test work. Give detailed explanation.

(g) Determine whether the series $\sum_{n=1}^{\infty} \ln\left(\dfrac{3n}{n+1}\right)$ converges or diverges. Make sure to state which test/s you use and provide justifying computations and arguments.

Solution: The series diverges by the Divergence Test. Another test that would work is the limit comparison test.

(h) Determine whether the series $\sum_{n=1}^{\infty} \dfrac{(-1)^n}{5n+3}$ converges or diverges. Make sure to state which test/s you use and provide justifying computations and arguments.

Solution: Let $b_n = \dfrac{1}{5n+3}$. Then

$$b_{n+1} \le b_n \text{ for all } n \ge 1$$

and $\lim_{n\to\infty} b_n = 0$. Therefore the series converges by the Alternating Series Test.

(i) Determine the convergence of $\displaystyle\sum_{k=1}^{\infty} k \cos\left(\frac{\pi k + 1}{2k}\right)$.

Solution: The terms converge to $1 \neq 0$ (use L'Hospital's Rule once), so by the Divergence Test, this series is divergent.

(j) Determine the convergence of $\displaystyle\sum_{k=1}^{\infty} \left(\frac{2k}{5k+5} + \frac{1}{(4)^k}\right)$

Solution: The terms converge to $2/5 \neq 0$, so by the Divergence Test, this series is divergent.

(k) Determine the convergence of $\displaystyle\sum_{k=1}^{\infty} \frac{2^{4k+1}}{5^{2k-1}}$. If this series is convergent, compute its sum.

Solution: This geometric series has ratio 16/25. By the geometric series test, the series converges to 160/9.

(l) Determine the convergence of $\sum_{k=2}^{\infty} \frac{8^{3k+1}}{9^{2k-1}}$. If this series is convergent, compute its sum.

Solution: This geometric series has ratio $\frac{8^3}{9^2}$. By the geometric series test, the series diverges.

5. Review pg 4-5 of `https://egunawan.github.io/spring18/notes/notes4_4lhopitals_rule.pdf`

(a) Compute $\lim_{n\to\infty} \left(1+\frac{1}{n}\right)^{3n}$.

Solution: Use L'Hospital's rule to compute $\boxed{e^3}$.

(b) Compute $\lim_{n\to\infty} \left(1+\frac{1}{2n}\right)^{n}$.

Solution: Use L'Hospital's rule to compute $\boxed{\sqrt{e}}$

(c) Compute $\lim_{n\to\infty} \left(1+\frac{2}{n}\right)^{5n}$.

Solution: Use L'Hospital's rule to compute $\boxed{e^{10}}$

(d) Compute $\lim_{n\to\infty} n^2 e^{-n}$.

Solution: Use L'Hospital's rule to twice to get $\boxed{0}$.

(e) Compute $\lim_{n\to\infty} \frac{\ln n}{n}$.

Solution: Apply L'Hospital's rule once once to get $\boxed{0}$.

(f) Compute $\lim_{n\to\infty} \frac{n \sin n}{n^2+1}$.

Solution: Observe that

$$-\frac{n}{n^2+1} \leq \frac{n \sin n}{n^2+1} \leq \frac{n}{n^2+1}.$$

Since $\lim_{n\to\infty} -\frac{n}{n^2+1} = 0 = \lim_{n\to\infty} \frac{n}{n^2+1}$, by the squeeze theorem we can conclude that $\lim_{n\to\infty} \frac{n \sin n}{n^2+1} = 0$.

6. Consider the series $\sum a_n = \sum_{n=2}^{\infty} \frac{\cos(n\pi)}{n-1}$.

(a) What are the first three terms in the series? ________________

Solution: $\frac{\cos(2\pi)}{2-1} = \frac{1}{1}$, $\frac{\cos(3\pi)}{3-1} = -\frac{1}{2}$, $\frac{\cos(4\pi)}{4-1} = \frac{1}{3}$

(b) Is the series convergent? You must justify. ________________

Solution: The series converges by the the alternating series test.

(c) Is the series $\sum a_n = \sum_{n=2}^{\infty} \frac{|\cos(n\pi)|}{n-1}$ convergent? You must justify. ______________

Solution: The series diverges by limit comparison test with $b_n = \frac{1}{\sqrt{n}}$ or by direct comparison test with $b_n = \frac{1}{n-1}$.

7. (a) Define an alternating series.

Solution: a series whose terms are alternately positive and negative.

(b) State the alternating series test.

Solution: see Sec 11.5, pg 732. (Given in fact sheet - make sure you know where it is).

(c) Determine whether the series

$$1 - \frac{1}{2} + \frac{1}{3} - \frac{1}{4} + \cdots = \sum_{k=1}^{\infty} \frac{(-1)^{n-1}}{n}$$

is convergent or divergent.

Solution: Example 1 Sec 11.5, pg 734

(d) Determine whether the series $\sum_{k=1}^{\infty} \frac{(-1)^n\ 3n}{4n-1}$ is convergent or divergent.

Solution: Example 2 Sec 11.5, pg 734

(e) Determine whether the series $\sum_{k=1}^{\infty} (-1)^{n+1} \frac{n^2}{n^3+1}$ converges or diverges.

Solution: Example 3 Sec 11.5, pg 734

8. (a) Evaluate $\lim_{n\to\infty} e^{-n}\sqrt{n}$.

Solution: Apply L'Hospital's rule once to compute $\boxed{0}$.

(b) Determine whether $\sum_{n=0}^{\infty} e^{-n}\sqrt{n}$ converges or diverges.

Solution: The series converges. Which test to use? You can use limit comparison test (compare the terms with any geometric sequence with ratio between $\frac{1}{e}$ and 1, for example, $b_n = \frac{1}{2^n}$ OR any p-sequence $b_n = \frac{1}{n^p}$ where $p > 1$).
You can also use the ratio test because you see an exponential factor e^{-n}.

(c) Evaluate $\lim_{n\to\infty} \frac{(\ln(n))^2}{n^2}$.

Solution: Apply L'Hospital's rule once to compute $\boxed{0}$.

(d) Determine whether $\displaystyle\sum_{n=1}^{\infty} \frac{(\ln(n))^2}{n^2}$ converges or diverges.

Solution: The series converges.

You can use limit comparison test (compare the terms with any p-sequence $b_n = \dfrac{1}{n^p}$ where $1 < p < 2$).
The Ratio Test does not work because you only see logarithmic and polynomial factors in the terms.

(e) Suppose $\displaystyle\sum_{n=1}^{\infty} a_n$ is a series with the property that

$$a_1 + a_2 + \cdots + a_n = 2 - 3(0.8)^n.$$

State whether $\sum_{n=1}^{\infty} a_n$ converges or diverges. If it converges, find its sum.

Solution: The expression above is the nth partial sum

$$S_n = 2 - 3(0.8)^n.$$

By definition, the series converges to

$$\lim_{n\to\infty} S_n = \boxed{2}.$$

9. Determine whether each series converges or diverges.

a.) $\displaystyle\sum_{n=1}^{\infty} \frac{\ln\ n}{n}$ *b.)* $\displaystyle\sum_{n=4}^{\infty} \frac{1}{2^n - 9}$ *i.)* $\displaystyle\sum_{n=1}^{\infty} \frac{2n^2 + 3n}{\sqrt{5 + n^5}}$ *ii.)* $\displaystyle\sum_{n=1}^{\infty} \frac{5}{2n^2 + 4n + 3}$

Solution: `https://egunawan.github.io/spring18/notes/hw11_4_to_worksheet_key.pdf`
More practice examples of series with only positive terms: `https://egunawan.github.io/spring18/notes/notes11_strategy_pos_terms_practice.pdf`