Math 1152, Spring 2018 — Exam 3 Fact Sheet

Useful trig facts.

$$\sin^2\theta + \cos^2\theta = 1, \quad \tan^2\theta + 1 = \sec^2\theta, \quad \cos^2\theta = \frac{1}{2}(1+\cos 2\theta), \quad \sin^2\theta = \frac{1}{2}(1-\cos 2\theta)$$

$$\sin\frac{\pi}{6} = \frac{1}{2}, \quad \sin\frac{\pi}{3} = \frac{\sqrt{3}}{2}, \quad \cos\frac{\pi}{6} = \frac{\sqrt{3}}{2}, \quad \cos\frac{\pi}{3} = \frac{1}{2}, \quad \sin\frac{\pi}{4} = \cos\frac{\pi}{4} = \frac{\sqrt{2}}{2}$$

Some derivatives. $\frac{d}{dx}b^x = \ln(b)b^x$, $\frac{d}{dx}\sin(x) = \cos(x)$, $\frac{d}{dx}\cos(x) = -\sin(x)$, $\frac{d}{dx}\tan(x) = (\sec(x))^2$, $\frac{d}{dx}\csc(x) = -\csc(x)\cot(x)$, $\frac{d}{dx}\sec(x) = \sec(x)\tan(x)$, $\frac{d}{dx}\cot(x) = -(\csc(x))^2$

L'Hôpital's rule. Suppose f, g are differentiable functions and $\lim_{x\to\infty} f(x)$ and $\lim_{x\to\infty} g(x)$ are both 0 or both $\pm\infty$. Then

$$\lim_{x\to\infty}\frac{f(x)}{g(x)} = \lim_{x\to\infty}\frac{f'(x)}{g'(x)}.$$

Geometric series. A series of the form $\sum ar^n$ is called a **geometric series**.

- if $|r| < 1$ then the series converges and $\sum_{n=0}^{\infty} ar^n = a/(1-r)$.
- if $|r| \geq 1$ then the series diverges.

Divergence test. Consider the series $\sum a_n$. If $\lim_{n\to\infty} a_n \neq 0$ then $\sum a_n$ **diverges**.

***p*-series test.** A series of the form $\sum_{n=1}^{\infty}\frac{1}{n^p}$ is called a p-series. The series converges if and only if $p > 1$.

Comparison test. Consider the series $\sum a_n$ with $a_n \geq 0$ for all n.

- if $a_n \leq b_n$ for all n and $\sum b_n$ converges, then $\sum a_n$ converges
- if $a_n \geq b_n$ for all n and $\sum b_n$ diverges, then $\sum a_n$ diverges

Limit comparison test. Consider the series $\sum a_n$ and $\sum b_n$ with $a_n, b_n \geq 0$ for all n and suppose that $\lim_{n\to\infty}\frac{a_n}{b_n} = c > 0$. Then $\sum a_n$ and $\sum b_n$ either both converge or both diverge.

Alternating series test. A series $\sum_{n=1}^{\infty}(-1)^{n+1}b_n$ where $b_n \geq 0$ for all n is called an alternating series. If

1. $b_{n+1} \leq b_n$ for all n large enough (ie. $\{b_n\}$ is an eventually decreasing sequence)
2. $\lim_{n\to\infty} b_n = 0$

then the series **converges**.

Ratio test. Consider the series $\sum_{n=1}^{\infty} a_n$ and suppose that $L = \lim_{n\to\infty}\left|\frac{a_{n+1}}{a_n}\right|$.

1. if $L < 1$ then $\sum_{n=1}^{\infty} a_n$ converges absolutely.
2. if $L > 1$ or $L = \infty$ then $\sum_{n=1}^{\infty} a_n$ diverges.
3. if $L = 1$ then the test is inconclusive.

Integral test. If f is continuous, non-negative, and decreasing on $[1,\infty)$ and $a_n = f(n)$, then

$$\sum_{n=1}^{\infty} a_n \text{ converges if and only if } \int_1^{\infty} f(x)\,dx \text{ converges.}$$

Integration by parts fomula. $\int u\,dv = uv - \int v\,du$

Power series coefficients. If $f(x) = \sum_{n=0}^{\infty} c_n(x-a)^n$, then $c_n = \dfrac{f^{(n)}(a)}{n!}$

Alternating Series Estimation Theorem. If $S := \sum_{k=r}^{\infty}(-1)^k b_k$, where $b_k > 0$, is the sum of an alternating series that satisfies

$$\text{(i) } b_{k+1} \leq b_k \quad \text{and} \quad \text{(ii) } \lim_{k\to\infty} b_k = 0,$$

then $|R_N| = |S - S_N| \leq b_{N+1}$, where $S_N := \sum_{k=r}^{N}(-1)^k b_k$.

Taylor's Inequality. If $|f^{n+1}(x)| \leq M$ for $|x-a| \leq d$, then the remainder $R_n(x)$ of the Taylor series satisfies the inequality

$$|R_n(x)| \leq \frac{M}{(n+1)!}|x-a|^{n+1} \text{ for } |x-a| \leq d$$

Table 1: Important Maclaurin Series and their Radii of Convergence

$$\frac{1}{1-x} = \sum_{n=0}^{\infty} x^n = 1 + x + x^2 + x^3 + \cdots \qquad R = 1$$

$$e^x = \sum_{n=0}^{\infty} \frac{x^n}{n!} = 1 + \frac{x}{1!} + \frac{x^2}{2!} + \frac{x^3}{3!} + \cdots \qquad R = \infty$$

$$\sin x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{(2n+1)!} = x - \frac{x^3}{3!} + \frac{x^5}{5!} - \frac{x^7}{7!} + \cdots \qquad R = \infty$$

$$\cos x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n}}{(2n)!} = 1 - \frac{x^2}{2!} + \frac{x^4}{4!} - \frac{x^6}{6!} + \cdots \qquad R = \infty$$

$$\tan^{-1} x = \sum_{n=0}^{\infty} (-1)^n \frac{x^{2n+1}}{2n+1} = x - \frac{x^3}{3} + \frac{x^5}{5} - \frac{x^7}{7} + \cdots \qquad R = 1$$

$$\ln(1+x) = \sum_{n=1}^{\infty} (-1)^{n-1} \frac{x^n}{n} = x - \frac{x^2}{2} + \frac{x^3}{3} - \frac{x^4}{4} + \cdots \qquad R = 1$$

$$(1+x)^k = \sum_{n=0}^{\infty} \binom{k}{n} x^n = 1 + kx + \frac{k(k-1)}{2!}x^2 + \frac{k(k-1)(k-2)}{3!}x^3 + \cdots \qquad R = 1$$

Tangents and areas. Suppose f and g are differentiable functions. Consider the curve defined by the parametric equations

$$x = f(t), \qquad y = g(t),$$

where y is a differentiable function of x. Then

$$\frac{dy}{dx} = \frac{\left(\dfrac{dy}{dt}\right)}{\left(\dfrac{dx}{dt}\right)} \quad \text{if} \quad \frac{dx}{dt} \neq 0.$$

The area under the curve from $x = a$ to $x = b$ which is traced out *once* by the curve, $\alpha \leq t \leq \beta$, can be calculated as follows:

$$\int_a^b \text{y dx} = \int_{\alpha}^{\beta} g(t)\ f'(t)\ \text{dt} \qquad \text{or} \qquad \int_a^b \text{y dx} = \int_{\beta}^{\alpha} g(t)\ f'(t)\ \text{dt}.$$