Suggestion: practice just a few problems from each topic.

Contents

1 11.8 power series

1. What is a power series?
2. What is the radius of convergence of a power series? What are the different possibilities?
3. In most cases, how do you find the radius of convergence of a power series?
4. From textbook: Find the radius of convergence and interval of convergence of the following series

$$\text{(a.)} \sum_{n=0}^{\infty} \frac{n(x+2)^n}{3^{n+1}}. \quad \text{(b.)} \sum_{n=0}^{\infty} n!x^{2n}. \quad \text{(c.)} \sum_{n=0}^{\infty} \frac{(x-3)^n}{n^5}. \quad \text{(d.)} \sum_{n=0}^{\infty} \frac{(x-3)^n}{n!}.$$

5. From WebAssign: Find the radius R and interval I of convergence of each series.

$$\text{(A.)} \sum_{n=1}^{\infty} \frac{x^n}{6n-1}. \quad \text{(B.)} \sum_{n=1}^{\infty} \frac{6^n(x+7)^n}{\sqrt{n}}. \quad \text{(C.)} \sum_{n=1}^{\infty} (-1)^n \frac{x^{n+1}}{n+4}. \quad \text{(D.)} \sum_{n=0}^{\infty} (-1)^n \frac{x^{5n}}{(2n)!}.$$

6. (a) Suppose that the radius of convergence of the power series $\sum c_n x^n$ is 16. What is the radius of convergence of the power series $\sum c_n x^{4n}$?
   (b) Suppose that the radius of convergence of the power series $\sum c_n x^n$ is R. What is the radius of convergence of the power series $\sum c_n x^{5n}$?
7. Determine the radius of convergence and interval of convergence for $\displaystyle\sum_{n=1}^{\infty} \frac{2^n}{n} x^n$.

2 11.9 power series: using geometric series and step-by-step integration/differenti

8. For each function, find a power series representation and determine the *interval* of convergence.
   (You can check your work with WolframAlpha. Type "series representation of ...")
   (a) $f(x) = \dfrac{1}{3+x}$
   (b) $f(x) = \dfrac{x^3}{5+x}$

(c) $f(x) = \dfrac{x}{1+10x^2}$

9. For each function, find a power series representation. Determine the *radius* of convergence.
   (a) $f(x) = \dfrac{1}{(2+x)^2}$
   (b) $f(x) = \dfrac{1}{(2+x)^3}$
   (c) $f(x) = \dfrac{x}{(2+x)^3}$
   (d) $f(x) = \ln(1+x)$
   (e) $f(x) = arctan(x)$
   (f) $\displaystyle\int \frac{1}{1+x^7}$ dx
   (g) $\displaystyle\int \frac{x}{1-x^7}$ dx

10. (a) If the interval of convergence of a power series $\displaystyle\sum_{n=0}^{\infty} c_n x^n$ is [-9,11), what is the radius of convergence of the series $\displaystyle\sum_{n=1}^{\infty} n\, c_n x^{n-1}$? Why?

    (b) If the interval of convergence of a power series $\displaystyle\sum_{n=0}^{\infty} c_n x^n$ is [-9,11), what is the radius of convergence of the series $\displaystyle\sum_{n=0}^{\infty} \frac{c_n}{n+1} x^{n+1}$? Why?

11. Find a power series centered at $x = 0$ for the function $\dfrac{1}{2-5x}$ and find its interval of convergence.

3 11.10 Taylor series

Khan Academy Taylor, Maclaurin, and Power series online quizzes: `https://www.khanacademy.org/math/calculus-home/series-calc/challenge-exercises-series-calc/e/taylor-maclaurin-power-series-challenge`

Khan Academy Series estimation online quizzes: `https://www.khanacademy.org/math/calculus-home/series-calc/challenge-exercises-series-calc/e/series-estimatio`

12. i. *If* f has a power series representation at 4, that is, if $f(x) = \displaystyle\sum_{n=0}^{\infty} c_n (x-4)^n$ for $|x-4| < R$, then its coefficients are given by the formula $c_n =$ ____________________.

    ii. Circle all the true statements and cross out all the false statements, and justify.

    (a) If the series $\displaystyle\sum_{n=1}^{\infty} c_n x^n$ converges for $|x| < R$, then $\displaystyle\lim_{n\to\infty} c_n x^n = 0$ for $|x| < R$.

    (b) If the series $\displaystyle\sum_{n=1}^{\infty} c_n x^n$ diverges for $x = 5$, then $\displaystyle\lim_{n\to\infty} c_n x^n \neq 0$ for $x = 5$.

iii. Find the Maclaurin series for $f(x) = 6(1-x)^{-2}$. (You may assume that $f(x)$ has a power series expansion). Find the associated radius of convergence.

iv. Use a Maclaurin series given in this table `http://egunawan.github.io/spring18/quizzes/11_10_table01.pdf` (will be given) to obtain the Maclaurin series for the function $f(x) = 8e^x + e^{8x}$. Find the radius of convergence.

v. Evaluate the indefinite integral $\left(8\int \frac{e^x - 1}{5x}\,\mathrm{dx}\right)$ as an infinite series.

vi. Find the Maclaurin series for $f(x) = e^{-4x}$ using the definition of a Maclaurin series. (You may assume that $f(x)$ has a power series expansion). Find the associated radius of convergence R.

13. (a) Use Table 1 to show that $\frac{d}{dx}\cos(x) = -\sin(x)$.

(b) Write the first 3 nonzero terms of the Maclaurin series for $\tan(x)$ using Table 1 and long division of power series.

(c) Use the series that you just computed for $\tan(x)$ to evaluate

$$\lim_{x\to 0} \frac{\tan(x) - x}{x^3}.$$

(d) Use a different method to evaluate $\lim_{x\to 0} \frac{\tan(x) - x}{x^3}$.

(e) Write the first 3 nonzero terms of the Maclaurin series for $e^x \sin(x)$ using Table 1 and multiplication of power series.

(f) Write the first 3 nonzero terms of the Maclaurin series for $\sec(x)$ using long division of power series and Table 1.

14. (a) An application of the Alternating Series Estimation Theorem is a way to ensure that we can get an approximation to the *definite* integral $\int_0^1 e^{-x^2}\,\mathrm{dx}$ using series so that the approximation is within a certain error bound (for example 1/1000).

T **F**

(b) We can always use the Alternating Series Estimation Theorem to ensure that we can get an approximation of a *function* using its Taylor polynomial so that the approximation is within a certain error bound (for example, 1/1000) on a certain interval. **T** **F**

15. (a) True or False? If $f(x) = 1 + 3x - 2x^2 + 5x^3 + \ldots$ for $|x| < 1$ then $f'''(0) = 30$.

(b) Can you write a Maclaurin series for $f(x) = \sqrt[3]{x}$? Explain why or why not.

16. Using Table 1 (series), prove that $Re^{i\theta} = R\cos\theta + iR\sin\theta$

17. True or False, with justification.

(a) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} = \frac{1}{e}$ **T** **F**

(b) $\displaystyle\sum_{n=0}^{\infty} \frac{(-1)^n}{n!} = -e$ **T** **F**

(c) If $f(x) = 2x - x^2 + \frac{1}{3}x^3 - \dots$ converges for all x, then $f'''(0) = 2$ **T** **F**

18. (a) Find the 3rd degree Taylor polynomial of the function $f(x) = xe^{-2x}$ centered at 0. Sketch this polynomial. Label all the important points - label at least four convenient points.

(b) (i) Approximate $f(x) = \ln(1 + 2x)$ by a Taylor polynomial of degree 3 centered at 1.

(ii) Use Taylor's Inequality to estimate the accurate of the approximation $T_3(x)$ of $f(x)$ when $0.8 \le x \le 1.2$. You do not need to simplify your answer.

(c) Approximate $f(x) = e^{4x^2}$ by a Taylor polynomial with degree 3 centered at 0.

19. Compute the Taylor series for $f(x) = \ln(x)$ at $a = 10$.

20. Find the Taylor series for $f(x) = \sqrt{x}$ centered at 9.

21. Determine the 2nd-degree Taylor polynomial $T_2(x)$ for $\arctan x$ at $a = 1$ and use Taylor's inequality to bound $|R_2(x)|$ if $|x - 1| \le \frac{1}{2}$, where $\arctan x = T_2(x) + R_2(x)$.

4 10.1-10.2 Calculus with parametric equations

22. For the following two parametric curves

$$(1)\ x = \cos t,\ y = \sin t \text{ for } 0 \le t \le 2\pi, \quad (2)\ x = -\sin(2t),\ y = -\cos(2t) \qquad \text{for} \qquad 0 \le t \le \frac{3\pi}{2}$$

eliminate the parameter to obtain an equation for the curve that directly relates x and y (non-parametric form of the curve) and then sketch the curve with an arrow indicating the direction it is traced out as t increases. Find the initial and final points.

23. i. (a) Find *parametric* equations for the top half of the circle centered at $(2, 3)$ with radius 5, oriented *clockwise*.

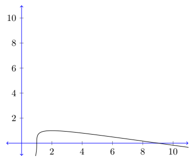
(b) Eliminate the parameter to find a Cartesian equation of the curve.

ii. Consider the curve described by the parametric equations

$$x = t^3 + 1$$

$$y = 2t - t^2, \qquad \text{for } -\infty < t < \infty$$

(a) Mark the orientation on the curve (direction of increasing values of t).



(b) Find the area enclosed by the x-axis and the given curve.

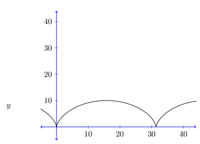
(c) Perform and describe a reality check by comparing your answer and the graph which has been drawn to scale.

iii. Consider the cycloid which is described by the parametric equations

$$x = 5(t - \sin t)$$

$$y = 5(1 - \cos t), \quad \text{for } \infty < t < \infty$$

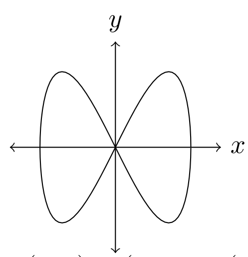
(a) Mark the orientation on the curve (direction of increasing values of t).



(b) Find the area enclosed by the x-axis and *one* arch of the cycloid. Hint: $dx = 5(1 - \cos t)\ dt$.

(c) Perform a reality check by comparing your answer and the graph (which is drawn to scale).

24. On the parametric curve $(x, y) = (\cos(t), \sin(2t))$, whose graph is below, determine (a) the slopes of the two tangent lines at the origin and (b) coordinates of the point in the first quadrant where the tangent line has slope -2.



$(x, y) = (\cos t, \sin(2t))$

5 10.3-10.4 Polar equations, sketch, derivatives and area

25. (a) Sketch the polar equation $r = \frac{5}{2}$

(b) Sketch the polar equation $\theta = \frac{\pi}{4}$

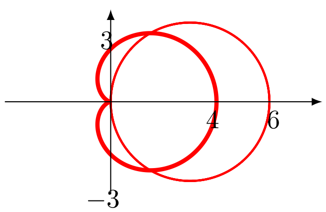
(c) Convert the polar equation $r = 3$ to Cartesian.

(d) Convert the polar equation $\theta = \frac{\pi}{3}$.

(e) Convert the polar equation $\theta = \frac{\pi}{6}$.

(f) Convert the polar equations $r = 9\cos\theta$ to Cartesian.

26. Consider the circle $r = 6\cos\theta$ and the cardioid $r = 2 + 2\cos\theta$.

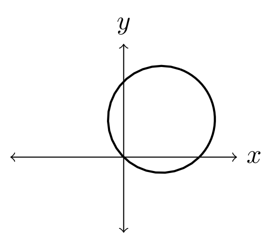


(a) Mark points on *both* curves where $\theta = 0, \dfrac{\pi}{4}$, and $\dfrac{\pi}{2}$.

(b) Shade in the area inside the circle and outside the cardioid.

(c) Find the area (which you shade) inside the circle and outside the cardioid.

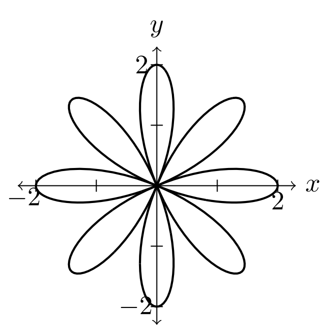
27. Below is the plot of the polar equation $r = \sin\theta + \cos\theta$.



Fill in the table below, use it to determine the orientation of the curve (direction of increasing θ), and find the equation of the tangent line to the curve at $(x, y) = (0, 0)$.

θ	$\sin\theta + \cos\theta$	(r, θ)	(x, y)
0			
$\pi/4$			
$\pi/2$			
π			

28. Below is a graph of $r = 2\cos 4\theta$. Determine the area enclosed by it.



6 9.1 Modeling with differential equations

29. (WebAssign 9.1 differential equations)

(a) For what values of k does the function $y = \cos(kt)$ satisfy the differential equation $4y'' = -9y$?

(b) Circle all functions which are solutions to $4y'' = -9y$. (Possibly none or all).

1. $y = -\cos(\frac{3t}{2})$
2. $y = \cos(\frac{3t}{2}) + 1$
3. $y = \sin(\frac{3t}{2})$
4. $y = \sin(\frac{3t}{2}) + \cos(\frac{3t}{2})$

(c) True or false? Every member of the family of functions $y = \dfrac{4\ln(x) + C}{x}$ is a solution of the differential equation

$$x^2y' + xy = 4$$

(d) Find a solution of the differential equation $x^2y' + xy = 4$ that satisfies the initial condition $y(1) = 2$.

(e) Find a solution of the differential equation $x^2y' + xy = 4$ that satisfies the initial condition $y(2) = 1$.

(f) Find a solution of the differential equation $x^2y' + xy = 4$ that satisfies the initial condition $y(3) = 1$.

(g) What can you say about a solution of the differential equation $y' = -\frac{1}{2}y^2$ *just* by looking at the differential equation? Circle all possibilities.

1. The function y must be equal to 0 on any interval on which it is defined.
2. The function y must be strictly increasing on any interval on which it is defined.
3. The function y must be increasing (or equal to 0) on any interval on which it is defined.
4. The function y must be decreasing (or equal to 0) on any interval on which it is defined.
5. The function y must be strictly decreasing on any interval on which it is defined.

(h) Verify that all members of the family $y = \frac{2}{x+C}$ are solutions of the differential equation $y' = -\frac{1}{2}y^2$.

(i) Write a solution of the differential equation $y' = -\frac{1}{2}y^2$ that is not a member of the family $y = \frac{2}{x+C}$.

(j) Find a solution of the initial-value problem. $y' = -\frac{1}{2}y^2 \qquad y(0) = 0.1$

(k) Find a solution of the initial-value problem. $y' = -\frac{1}{4}y^2 \qquad y(0) = 0.2$

(l) Find a solution of the initial-value problem. $y' = -\frac{1}{3}y^2 \qquad y(0) = 0.5$

(m) Find a solution of the initial-value problem. $y' = -\frac{1}{6}y^2 \qquad y(0) = 0.5$

(n) A population is modeled by the differential equation

$$\frac{dP}{dt} = 1.1P\left(1 - \frac{P}{4000}\right)$$

1. For what values of P is the population increasing?
2. For what values of P is the population decreasing?
3. What are the equilibrium solutions?

(o) A function $y(t)$ satisfies the differential equation

$$\frac{dy}{dt} = y^4 - 8y^3 + 15y^2.$$

1. What are the constant solutions of the equation?
2. Sketch the polynomial $t^4 - 8t^3 + 15t^2$. In particular, mark the x-intercepts.
3. For what values of y is y increasing?
4. For what values of y is y decreasing?

30. (a) True or false? Every differential equation has a constant solution. (If T, explain. If F, give a counterexample.)

(b) Consider the differential equation $\frac{dy}{dt} = 5 - 2y$.

i. Find all constant solution/s.

ii. Which of the following is a family of solutions? You may need to circle more than one.

$$y(t) = 1 + Ke^{-2t} \qquad y(t) = -Ke^{-2t} \qquad y(t) = \frac{5}{2} + Ke^{-2t} \qquad y(t) = \frac{5}{2} - Ke^{-2t}$$

iii. Which of the functions below satisfy the differential equation $y'' + y = \sin x$?

(a) $y = \sin x$

(b) $y = \cos x$

(c) $y = \frac{1}{2}x\sin x$

(d) $y = -\frac{1}{2}x\cos x$

31. (a) Draw a rough sketch of a possible solution to the logistic differential equation $\frac{dP}{dt} = 5P\left(1 - \frac{P}{8}\right)$. **You do not need to solve this differential equation to draw a rough sketch**.

7 9.3 Separable differential equations

32. (a) Find the solution of the differential equation that satisfies the given initial condition.

$$\frac{dy}{dx} = \frac{x}{y}, \quad y(0) = -9$$

(b) Find the solution of the differential equation that satisfies the given initial condition.

$$xy' + y = y^2, \quad y(1) = -8$$

(c) Consider the differential equation $(x^2 + 15)y' = xy$.

i. Find all constant solutions.

ii. Find all solutions.

(d) A tank contains 500 L of brine with 15 kg of dissolved salt. Brine having .2 kg of salt per liter of water enters the tank at a rate of 10 L/min. The solution is kept thoroughly mixed and is drained from the tank at 10 L/min. How much salt is in the tank after t minutes? After 20 min? In the long run?

(e) The differential equation below models the temperature of a 86^o C cup of coffee in a 20^o C room, where it is known that the coffee cools at a rate of 1^o C per minute when its temperature is 70^o C. Solve the differential equation to find an expression for the temperature of the coffee at time t. (Let y be the temperature of the cup of coffee in ^oC, and let t be the time in minutes, with $t = 0$ corresponding to the time when the temperature was 86^o C.)

$$\frac{dy}{dt} = -\frac{1}{50}(y - 20)$$

(f) A tank contains 8000 L of brine with 14 kg of dissolved salt. Pure water enters the tank at a rate of 80 L/min. The solution is kept thoroughly mixed and drains from the tank at the same rate.

1. How much salt is in the tank after t minutes?
2. How much salt is in the tank after 20 minutes?

(g) Find the orthogonal trajectories of the family of curves $y^2 = 8kx^3$. Sketch these orthogonal trajectories.