University of Connecticut
Department of Mathematics

MATH 1152Q EXAM 1 PRACTICE (KEY) SPRING 2018

Read This First !

- From each number, choose a couple parts that seem the most challenging.
- **You will earn a small amount of bonus 'style points' for a legible, coherent, and non-ambiguous paper. Your reader should not need to reread your solution several times to find a train of thought. In addition, you should use correct mathematical notations. This includes not writing an equal sign between two unequal objects, not treating the symbol ∞ like a number, and not attempting to multiply 0 with the symbol ∞.**
- Please read each question carefully. Show **ALL** work clearly in the space provided. In order to receive full credit on a problem, solution methods must be complete, logical and understandable.
- Answers must be clearly labeled in the spaces provided after each question. Please cross out or fully erase any work that you do not want graded. The point value of each question is indicated after its statement. No books or other references are permitted.
- All technology (phones, calculators) and books/ notes should be placed inside your bag.

1. Extra credit: know the correct spelling of everyone's name and be able to identify everyone in the classroom.
2. For the following questions, circle TRUE or FALSE, and give a justification. True statements should be argued for using facts, theorems or definitions from class.

(a) If $\lim_{n\to\infty} a_n = 0$ then the series $\sum a_n$ converges. **T** **F**

Justification:
Answer: False. Counterexample: $\dfrac{1}{\sqrt{n}} \to 0$ as $n \to \infty$, but $\sum \dfrac{1}{\sqrt{n}}$ diverges.

(b) If $a_n > 0$, $b_n > 0$ for all n, $\sum b_n$ diverges, and $\lim_{n\to\infty} \dfrac{a_n}{b_n} = 0$, then $\sum a_n$ diverges. **T** **F**

Justification:
Answer: False. Counterexample: Let $b_n = \dfrac{1}{n}$ (and so $\sum b_n$ by p-series/harmonic series test) and $a_n = \dfrac{1}{n^2}$. We have $\lim_{n\to\infty} \dfrac{a_n}{b_n} = \lim_{n\to\infty} \dfrac{n}{n^2} = 0$. But $\sum a_n$ converges.

(c) If $a_n > 0$ for all n & $\lim_{n\to\infty} \dfrac{a_{n+1}}{a_n} = 0$, then $\sum a_n$ is convergent by the ratio test **T** **F**

Justification:
Answer: True because $0 < 1$.

(d) If a_n and b_n are both positive for all n and $\lim_{n\to\infty} \dfrac{a_n}{b_n} = 0$, then $\sum a_n$ is convergent by the limit comparison test **T** **F**

Justification:
Answer: False. You cannot conclude this in general (for example, when you don't know that $\sum b_n$ converges). Counterexample: $a_n = n$ and $b_n = n^2$.

(e) The harmonic series $\sum 1/n$ is convergent by the p-series test **T** **F**

Justification:
Answer: False. The harmonic series is divergent.

(f) We can use the ratio test *alone* to show the geometric series $\sum \dfrac{2^n}{3^n}$ converges **T** **F**

Justification:
Answer: True. The ratio of $\dfrac{2^{n+1}}{3^{n+1}} \dfrac{3^n}{2^n}$ goes to $\dfrac{2}{3} < 1$ as $n \to \infty$.

(g) We can use the p-series test *alone* to show the series $\sum 2^n/3^n$ converges **T** **F**
Justification:
Answer: False because $\sum 2^n/3^n$ does not look like a p-series.

(h) We can apply the monotonic sequence theorem to show that the geometric **sequence** $\{2^n/3^n\}_{n=1}^{\infty}$ is convergent **T** **F**
Justification:
Answer: True. The sequence $\left\{\left(\frac{2}{3}\right)^n\right\}_{n=1}^{\infty}$ is bounded below (for example, by 0 and -2) and bounded above (for example, by $\frac{2}{3}$ and 5). This sequence is also decreasing. By the monotonic sequence theorem, the sequence is convergent.

(i) We can apply the monotonic sequence theorem to show that the harmonic sequence $\left\{\frac{1}{n}\right\}_{n=1}^{\infty}$ is convergent **T** **F**
Justification:
Answer: True. The sequence $\left\{\frac{1}{n}\right\}_{n=1}^{\infty}$ is bounded below (for example, by 0 and -2) and bounded above (for example, by 1 and 5). This sequence is also decreasing. By the monotonic sequence theorem, the sequence is convergent.

(j) We can apply the squeeze theorem to show that the alternating harmonic sequence $\left\{\frac{(-1)^n}{n}\right\}$ is convergent **T** **F**
Justification:
Answer: True. Squeeze each term between $\left\{-\frac{1}{n}\right\}$ and $\left\{\frac{1}{n}\right\}$.

(k) It is impossible for a subset of a line to have infinitely many points and have length zero. **T** **F**
Justification:
Answer: F. It is possible, for example, the Cantor set, see `http://egunawan.github.io/spring18/hw/problemset_a_s18.pdf`

(l) The divergence of the p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ for $0 < p < 1$ follows from divergence of the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ and the comparison test. **T** **F**
Justification:
Answer: T. Let $a_n = \frac{1}{n^p}$ (for some $0 < p < 1$) and $b_n = \frac{1}{n^1}$. Then $a_n > b_n$ for all $n = 2, 3, \ldots$. Since $\sum b_n$ diverges (by divergence of the harmonic series), by the comparison test $\sum a_n$ also diverges.

(m) The convergence of the p-series $\sum_{n=1}^{\infty} \frac{1}{n^p}$ for $p > 1$ follows from divergence of the harmonic series $\sum_{n=1}^{\infty} \frac{1}{n}$ and the comparison test. **T** **F**
Justification:
Answer: F. You can't conclude the convergence of a series by comparing it with a divergent series.

(n) Convergence of $\sum \frac{1}{n^p}$ for $p > 1$ can be shown with the ratio test. **T** **F**
Justification:
Answer: False. Since $\frac{a_{n+1}}{a_n} = \frac{n^p}{(n+1)^p} \to 1$ as $n \to \infty$, the ratio test is inconclusive for this series.

(o) Divergence of a p-series for $p < 1$ can be shown with the ratio test. **T** **F**
Justification:
Answer: False. Since $\lim_{n\to\infty} \frac{a_{n+1}}{a_n} = \lim_{n\to\infty} \frac{n^p}{(n+1)^p} = 1$, the ratio test is inconclusive for this series.

3. (a) State the contrapositive of the factual statement: "If the sequence $\{a_n\}$ is unbounded, then it is divergent".
Answer: If the sequence $\{a_n\}$ is convergent, then it is bounded.

(b) Is the contrapositive statement you wrote as your answer to part (a) true or false? **Justification (explain or give a counterexample):**
Answer: True. Since the statement in part (a) is true (even though I haven't proven it), the contrapositive statement is also true (because the contrapositive statement is equivalent to the original statement).

(c) The converse of part (a) is the following: "If the sequence $\{a_n\}$ is divergent, then $\{a_n\}$ is unbounded". Is this true or false? **Justification (explain or give a counterexample):**
Answer: False. Counterexample: let $a_n = (-1)^n$.

4. Answer the following on the line provided.

(a) What is the 100th term of the sequence $\{2, 5, 8, 11, \dots\}$?
(The terms 2 and 5 are the first and second term, respectively) ______________

Answer: $\boxed{-1 + 300 = 299}$

(b) Find a formula for the general term a_n of the sequence $\left\{1, -\frac{2}{5}, \frac{3}{25}, -\frac{4}{125}, \frac{5}{625} \dots \right\}$. Make sure to specify your starting value of n.
Answer: Observe that for $n \geq 1$,

$$a_n = \frac{n}{(-5)^{n-1}}$$

(c) Write the geometric series $4 + 2 + 1 + \frac{1}{2} \cdots$ in standard form.
(summation notation $\sum$) ______________

Answer:

$$\sum_{n=-2}^{\infty} \left(\frac{1}{2}\right)^n \text{ or } \sum_{n=-1}^{\infty} \left(\frac{1}{2}\right)^{n-1} \text{ or } \sum_{n=0}^{\infty} \left(\frac{1}{2}\right)^{n-2}$$

(d) Find the 4th term a_4 in the recursive sequence $a_{n+1} = 2a_n + a_{n-1}$
when $a_1 = 1$ and $a_2 = 1$. ______________

Answer: $\boxed{a_4 = 7}$ because $a_3 = 2a_2 + a_1 = 2\,.\,1 + 1 = 3$, so $a_4 = 2a_3 + a_2 = 2\,.3 + 1 = 7$.

(e) Find the 7th term a_7 in the recursive sequence $a_{n+1} = a_n + a_{n-1}$
when $a_1 = 2$ and $a_2 = 3$. ______________

Answer: $\boxed{a_7 = 34}$

(f) We can use geometric series to compute

$$\frac{1}{10} + \frac{1}{100} + \frac{1}{1000} + \dots.$$

What fraction is this equal to?

Answer: $\boxed{\frac{1}{9}.}$

(g) We can use geometric series to compute $0.9999\dots$. What fraction is this equal to?

Answer: $\boxed{\frac{1}{1}.}$

(h) One of the two decimal expansions for a number is $2.449999\dots$. What's the other?
Answer: $\boxed{2.45}$ See `https://egunawan.github.io/spring18/hw/problemset_a_s18.pdf`

(i) Use geometric series to compute the fraction for $1.833333\dots$
Answer: $\boxed{11/6}$
Perform a sanity check against your answer.
Answer: For example, you can check that your answer is bigger than 1.5 but smaller than 2.

(j) Use geometric series to compute the fraction for $1.08333333333333333\dots$
Answer: $\boxed{13/12}$
Perform a sanity check against your answer.
Answer: For example, you can check that your answer is bigger than 1 but smaller than 1.1.

(k) Does the series $\displaystyle\sum_{n=1}^{\infty} -\ln\left(\frac{n}{2n+7}\right)$ converge or diverge? ______________

Answer: The series <u>diverges</u>. You can apply divergence test because $\displaystyle\lim_{n\to\infty} -\ln\left(\frac{n}{2n+7}\right) = -\ln\left(\frac{1}{2}\right) \neq 0$.

(l) Find the sum of the series $\displaystyle\sum_{k=0}^{\infty} 5\left(\frac{2}{3}\right)^k$. ______________

Answer: $\boxed{15}$

(m) Find the sum of the series $\displaystyle\sum_{n=2}^{\infty} 5\left(\frac{(-6)^{n-1}}{7^n}\right)$. ________________

Answer: $\boxed{-\dfrac{30}{91}.}$

Perform a sanity check against your answer.

Answer: For example, since the first term of the series is a negative number, you can check that your answer is negative.

(n) Find the sum of the series ________________

$$-5+3-\frac{9}{5}+\frac{27}{25}-\frac{81}{125}+\ldots.$$

Answer: *If* you want your first term to be ar^0 (and, consequently, your second term to be ar^1), then you would write $r=-\dfrac{3}{5}$ with $a=-5$. Therefore the sum of the infinite series is

$$\boxed{-5\frac{1}{1-r}=-5\frac{1}{1+\frac{3}{5}}=-\frac{25}{8}}.$$

Perform a sanity check against your answer.

Answer: For example, since the first term of the series is -5, you can check that your answer is negative.

(o) Write an expression for the nth term in the sequence ________________
$\left\{\dfrac{1}{2},\dfrac{1}{6},\dfrac{1}{24},\dfrac{1}{120},\ldots\right\}$. (The terms $\dfrac{1}{2}$ and $\dfrac{1}{6}$ are the first and second terms in the sequence)

Answer: $\boxed{a_n=\dfrac{1}{(n+1)!} \text{ for } n=1,2,3,\ldots}$

(p) Write an equivalent series with index summation beginning at $n=0$. ________________
$\displaystyle\sum_{n=2}^{\infty}\frac{2^n}{(n-2)!}$

Answer: $\boxed{\displaystyle\sum_{n=0}^{\infty}\frac{2^{n+2}}{(n)!}}$

Perform a sanity check against your answer.

Answer: For example, you can write down the first two terms of the series and confirm that they match.

(q) Write an equivalent series with index summation beginning at $n=1$. ________________
$\displaystyle\sum_{n=2}^{\infty}\frac{2^n}{(n-2)!}$

Answer: $\boxed{\displaystyle\sum_{n=1}^{\infty}\frac{2^{n+1}}{(n-1)!}}$

Perform a sanity check against your answer.

Answer: For example, you can write down the first two terms of the series and confirm that they match.

(r) For what values of k does the series $\sum\dfrac{5}{n^k}$ converge? Please explain. ________________

Answer: $\boxed{k>1, \text{ by the } p\text{-series test.}}$

(s) Find the values of A so that the series $\displaystyle\sum_{n=1}^{\infty}\frac{(A)^{n-1}}{3^{n-1}}$ is convergent. Please explain.

Answer: By the geometric series theorem/test, we need $\left|\dfrac{A}{3}\right|<1$, so $\boxed{-3<A<3}$.

(t) Find the values of B so that the series $\displaystyle\sum_{n=1}^{\infty}\frac{(B-3)^{n-1}}{3^{n-1}}$ is convergent. Please explain.

Answer: By the geometric series theorem/test, we need $\left|\dfrac{B-3}{3}\right|<1$, so $\boxed{0<B<6}$.

(u) Find the values of C so that the series $\displaystyle\sum_{n=5}^{\infty} \frac{(C-2)^n}{3^{n+1}}$ is convergent. Please explain.

Answer: By the geometric series theorem/test, we need $\left|\frac{C-2}{3}\right| < 1$, so $\boxed{-1 < C < 5}$.

5. (a) (copy from Sec 11.1 page 696) Let $\{a_n\}$ be a sequence and let $L \in \mathbb{R}$ (this notation means that L is a real number). What does $\lim\limits_{n\to\infty} a_n = L$ mean?

(b) (copy from Sec 11.1 page 696) Let $\{a_n\}$ be a sequence and let $L \in \mathbb{R}$ (this notation means that L is a real number). What does $\lim\limits_{n\to\infty} a_n = L$ mean? Use the $\epsilon - N$ definition.

(c) (copy from Sec 11.2, page 708) Let $\{c_n\}$ be a sequence. What is a partial sum of $\{c_n\}$?

(d) (copy from Sec 11.2, page 708) Let $\{c_n\}_{n=1}^{\infty}$ be a sequence. We say that the infinite series $\sum_{n=1}^{\infty} c_n$ is *convergent* if

____. (Hint: your answer should include the words 'limit' and 'partial sums') If the above blank is not true, then we say that $\sum_{n=1}^{\infty} c_n$ is *not convergent* or *divergent*.

(e) Let $a_k = \dfrac{5k^2-9}{k^2-4}$ for $k = 3, 4, 5, \ldots$. Prove that the sequence $\{a_k\}_{k=3}^{\infty}$ converges to 5 using the $\epsilon - N$ definition.
Answer: Suppose $\epsilon > 0$. I choose $N = \ldots$. Follow the boxed answers in `https://egunawan.github.io/spring18/notes/notes11_1choosingN.pdf`

(f) Let $a_k = \dfrac{1}{k^2+3}$ for k natural numbers. Prove that $\lim\limits_{k\to\infty} a_k = 0$ using the $\epsilon - N$ definition.

ANSWER (a possible answer): Let ϵ be a positive number. I choose $N = \sqrt{\frac{1}{\epsilon}}$. Then, if $k > N$, we have

$$\begin{aligned}
|a_k - L| &= \left|\frac{1}{k^2+3} - 0\right| \\
&= \frac{1}{k^2+3} \\
&< \frac{1}{N^2+3} \text{ since } k > N \text{ implies that } \frac{1}{k^2+3} < \frac{1}{N^2+3} \\
&= \frac{1}{\left(\sqrt{\frac{1}{\epsilon}}\right)^2 + 3} \text{ because } N = \sqrt{\frac{1}{\epsilon}} \\
&= \frac{1}{\left(\frac{1}{\epsilon}\right) + 3} \\
&< \frac{1}{\left(\frac{1}{\epsilon}\right)} \\
&= \epsilon.
\end{aligned}$$

(g) Let $a_k = \dfrac{3k+2}{2k-1}$ for $k = 1, 2, 3, \ldots$. Prove that $\lim a_k = \dfrac{3}{2}$ using the $\epsilon - N$ definition.

ANSWER (a possible answer): Let ϵ be a positive number. I choose $N = \dfrac{7}{4\epsilon} + \dfrac{1}{2}$. Then, if $k > N$, we have

$$\begin{aligned}
|a_k - L| &= \left|\frac{3k+2}{2k-1} - \frac{3}{2}\right| \\
&= \left|\frac{3k+2-\frac{3}{2}(2k-1)}{2k-1}\right| \\
&= \left|\frac{3k+2-3k+\frac{3}{2}}{2k-1}\right| \\
&= \left|\frac{\frac{7}{2}}{2k-1}\right| \\
&= \frac{\frac{7}{2}}{2k-1} \quad \text{because } k \geq 1, \text{ so } \frac{\frac{7}{2}}{2k-1} \text{ is positive} \\
&< \frac{\frac{7}{2}}{2N-1} \quad \text{because } k > N, \text{ so } \frac{1}{2k-1} < \frac{1}{2N-1} \\
&= \frac{\frac{7}{2}}{2\left(\frac{7}{4\epsilon} + \frac{1}{2}\right) - 1} \quad \text{since } N = \frac{7}{4\epsilon} + \frac{1}{2} \\
&= \frac{\frac{7}{2}}{\left(\frac{7}{2\epsilon} + 1\right) - 1} \\
&= \frac{\frac{7}{2}}{\left(\frac{7}{2\epsilon}\right)} \\
&= \frac{7}{2} \cdot \frac{2\,\epsilon}{7} \\
&= \epsilon.
\end{aligned}$$

(h) Let $a_k = \dfrac{k^2+2}{k^2-3}$ for k natural numbers. Show that $\{a_n\}$ converges to 1 using the ϵ,N definition.

ANSWER (a possible answer): Let ϵ be a positive number. I choose $N = \sqrt{\frac{5}{\epsilon} + 3}$ or $N = 2$, whichever is greater. Note that $N \geq 2$. Then, if $k > N$, we have

$$\begin{aligned}
|a_k - L| &= \left|\frac{k^2+2}{k^2-3} - 1\right| \\
&= \left|\frac{k^2+2-(k^2-3)}{k^2-3}\right| \\
&= \left|\frac{2+3}{k^2-3}\right| \\
&= \left|\frac{5}{k^2-3}\right| \\
&= \frac{5}{k^2-3} \quad \text{because } k > N \text{ and } N \geq 2, \text{ so } k \leq 2 \\
&< \frac{5}{N^2-3} \quad \text{since } k > N, \text{ so } \frac{1}{k^2-3} < \frac{1}{N^2-3} \\
&= \frac{5}{\left(\sqrt{\frac{5}{\epsilon}+3}\right)^2 - 3} \quad \text{because } N = \sqrt{\frac{1}{\epsilon}+3} \\
&= \frac{5}{\frac{5}{\epsilon}+3-3} \\
&= \frac{5}{\left(\frac{5}{\epsilon}\right)} \\
&= \epsilon.
\end{aligned}$$

ANSWER (another possible answer): Let ϵ be a positive number. I choose $N = \frac{5}{\epsilon} + 3$. Note that $N \geq 2$ since $\frac{5}{\epsilon} + 3 > 2$. Then, if $k > N$, we have

$$
\begin{aligned}
|a_k - L| &= \left|\frac{k^2+2}{k^2-3} - 1\right| \\
&= \left|\frac{k^2+2-(k^2-3)}{k^2-3}\right| \\
&= \left|\frac{2+3}{k^2-3}\right| \\
&= \left|\frac{5}{k^2-3}\right| \\
&= \frac{5}{k^2-3} \quad \text{because } k > N \text{ and } N \geq 2, \text{ so } k \leq 2 \\
&< \frac{5}{N^2-3} \quad \text{since } k > N, \text{ so } \frac{1}{k^2-3} < \frac{1}{N^2-3} \\
&= \frac{5}{\left(\frac{5}{\epsilon}+3\right)^2 - 3} \quad \text{because } N = \frac{1}{\epsilon} + 3 \\
&= \frac{5}{\left(\frac{25}{\epsilon^2} + \frac{30}{\epsilon} + 9\right) - 3} \quad \text{because } N = \frac{1}{\epsilon} + 3 \\
&= \frac{5}{\frac{25}{\epsilon^2} + \frac{30}{\epsilon} + 6} \\
&< \frac{5}{\left(\frac{30}{\epsilon}\right)} \quad \text{because } \frac{25}{\epsilon^2} + \frac{30}{\epsilon} + 6 \ > \ \frac{30}{\epsilon} \\
&= \frac{5}{\left(\frac{30}{\epsilon}\right)} \\
&= \frac{\epsilon}{6} \\
&< \epsilon.
\end{aligned}
$$

6.
   - Write the statement of the geometric series test/theorem as stated in Stewart Sec 11.2.
   - Write the statement of the divergence test as stated in Stewart Sec 11.2 (either box no. 6 or 7 is OK).
   - Write the statement of the limit comparison test as stated in Stewart Sec 11.4.
   - Write the statement of the ratio test for positive terms as stated in Stewart Sec 11.6.
   - Write the alternating series test/theorem as stated in Stewart Sec 11.5.

7. The following questions ask you to determine the converge/divergence of a series. To receive credit, give detailed explanation (for example, follow my answer key - to be posted).

   (a) Determine whether the series $\displaystyle\sum_{k=1}^{\infty} \frac{k^k}{7^k (k)!}$ is convergent.

   Answer: This series converges. You know ratio test will work because you see at least ONE of factorial, exponent, and n^n. See last page of class notes `https://egunawan.github.io/spring18/notes/notes11_6part3.pdf`

   (b) Determine whether the series $\displaystyle\sum_{k=1}^{\infty} \frac{k^k}{2^k (k)!}$ is convergent.

   Answer: In contrast to (the *very similar* series) above, this series diverges. You know ratio test will work because you see at least ONE of factorial, exponent, and n^n. See last pg of notes `https://egunawan.github.io/spring18/notes/notes11_6part3.pdf`

   (c) Determine whether the series $\displaystyle\sum_{n=3}^{\infty} \frac{6}{n\sqrt{n^2-8}}$ converges or diverges.

Answer: Let $a_n = \dfrac{6}{n\sqrt{n^2-8}}$. Let $b_n = \dfrac{1}{n^2}$.
Then $\dfrac{a_n}{b_n} \to 6$ (which is a positive number) as $n \to \infty$.
Since $\sum b_n$ converges by the p-series test, we conclude that $\sum a_n$ converges by limit comparison test.

(d) Determine whether the series $\sum_{n=1}^{\infty} \dfrac{\sqrt{n^3+1}}{3n^3-4n+2}$ converges or diverges. Make sure to state which test/s you use and provide justifying computations and arguments.
Answer: Let $b_n = \dfrac{1}{n^{3/2}}$. Note that $\sum b_n$ converges since it's a p-series with $p = 3/2 > 1$. Moreover,

$$\begin{aligned}\lim_{n\to\infty} \frac{a_n}{b_n} &= \lim_{n\to\infty} \frac{\sqrt{n^3+1}\quad n^{3/2}}{3n^3-4n+2} \\ &= \lim_{n\to\infty} \frac{\sqrt{n^3\ (n^3+1)}}{3n^3-4n+2} \\ &= \lim_{n\to\infty} \frac{\sqrt{n^6+n^3}}{3n^3-4n+2} \\ &= \frac{1}{3} > 0.\end{aligned}$$

Therefore, since $\sum b_n$ converges, $\sum a_n$ also converges by the Limit Comparison Test.

(e) Determine whether the series $\sum_{n=1}^{\infty} \dfrac{n!}{7^n(n+8)!}$ converges or diverges. Make sure to state which test/s you use and provide justifying computations and arguments.
Answer: The series converges. Both the ratio test and limit comparison test work. Make sure you give detailed explanation.

(f) Determine whether the series $\sum_{n=1}^{\infty} \dfrac{(n+8)!}{7^n n!}$ converges or diverges. Make sure to state which test/s you use and provide justifying computations and arguments. (This looks similar to above, but this isn't a typo).
Answer: The series converges. Both the ratio test and limit comparison test work. Give detailed explanation.

(g) Determine whether the series $\sum_{n=1}^{\infty} \ln\left(\dfrac{3n}{n+1}\right)$ converges or diverges. Make sure to state which test/s you use and provide justifying computations and arguments.
Answer: The series diverges by the Divergence Test. Another test that would work is the limit comparison test.

(h) Determine whether the series $\sum_{n=1}^{\infty} \dfrac{(-1)^n}{5n+3}$ converges or diverges. Make sure to state which test/s you use and provide justifying computations and arguments.
Answer: Let $b_n = \dfrac{1}{5n+3}$. Then

$$b_{n+1} \le b_n \text{ for all } n \ge 1$$

and $\lim_{n\to\infty} b_n = 0$. Therefore the series converges by the Alternating Series Test.

(i) Determine the convergence of $\displaystyle\sum_{k=1}^{\infty} k \cos\left(\frac{\pi k+1}{2k}\right)$.
Answer: The terms converge to $1 \neq 0$ (use L'Hospital's Rule once), so by the Divergence Test, this series is divergent.

(j) Determine the convergence of $\displaystyle\sum_{k=1}^{\infty} \left(\frac{2k}{5k+5} + \frac{1}{(4)^k}\right)$
Answer: The terms converge to $2/5 \neq 0$, so by the Divergence Test, this series is divergent.

(k) Determine the convergence of $\displaystyle\sum_{k=1}^{\infty} \frac{2^{4k+1}}{5^{2k-1}}$. If this series is convergent, compute its sum.
Answer: This geometric series has ratio $16/25$. By the geometric series test, the series converges to $160/9$.

(l) Determine the convergence of $\displaystyle\sum_{k=2}^{\infty} \frac{8^{3k+1}}{9^{2k-1}}$. If this series is convergent, compute its sum.
Answer: This geometric series has ratio $\dfrac{8^3}{9^2}$. By the geometric series test, the series diverges.

8. Review pg 4-5 of `https://egunawan.github.io/spring18/notes/notes4_4lhopitals_rule.pdf`

   (a) Compute $\lim_{n\to\infty}\left(1+\frac{1}{n}\right)^{3n}$.

   Answer: Use L'Hospital's rule to compute $\boxed{e^3}$.

   (b) Compute $\lim_{n\to\infty}\left(1+\frac{1}{2n}\right)^{n}$.

   Answer: Use L'Hospital's rule to compute $\boxed{\sqrt{e}}$

   (c) Compute $\lim_{n\to\infty}\left(1+\frac{2}{n}\right)^{5n}$.

   Answer: Use L'Hospital's rule to compute $\boxed{e^{10}}$

   (d) Compute $\lim_{n\to\infty} n^2e^{-n}$.

   Answer: Use L'Hospital's rule to twice to get $\boxed{0}$.

   (e) Compute $\lim_{n\to\infty}\frac{\ln n}{n}$.

   Answer: Apply L'Hospital's rule once once to get $\boxed{0}$.

   (f) Compute $\lim_{n\to\infty}\frac{n\sin n}{n^2+1}$.

   Answer: Observe that

$$-\frac{n}{n^2+1}\le\frac{n\sin n}{n^2+1}\le\frac{n}{n^2+1}.$$

   Since $\lim_{n\to\infty}-\frac{n}{n^2+1}=0=\lim_{n\to\infty}\frac{n}{n^2+1}$, by the squeeze theorem we can conclude that $\lim_{n\to\infty}\frac{n\sin n}{n^2+1}=0$.

9. Consider the series $\sum a_n=\sum_{n=2}^{\infty}\frac{\cos(n\pi)}{n-1}$.

   (a) What are the first three terms in the series? ________________

   Answer: $\frac{\cos(2\pi)}{2-1}=\frac{1}{1}$, $\frac{\cos(3\pi)}{3-1}=-\frac{1}{2}$, $\frac{\cos(4\pi)}{4-1}=\frac{1}{3}$

   (b) Is the series convergent? You must justify. ________________

   Answer: The series <u>converges</u> by the the alternating series test.

   (c) Is the series $\sum a_n=\sum_{n=2}^{\infty}\frac{|\cos(n\pi)|}{n-1}$ convergent? You must justify. ________________

   Answer: The series <u>diverges</u> by limit comparison test with $b_n=\frac{1}{\sqrt{n}}$ or by direct comparison test with $b_n=\frac{1}{n-1}$.

10. (a) Define an alternating series.

   Answer: a series whose terms are alternately positive and negative.

   (b) State the alternating series test.

   Answer: see Sec 11.5, pg 732.

   (c) Determine whether the series

$$1-\frac{1}{2}+\frac{1}{3}-\frac{1}{4}+\cdots=\sum_{k=1}^{\infty}\frac{(-1)^{n-1}}{n}$$

   is convergent or divergent.

   Answer: $\boxed{\text{Example 1 Sec 11.5, pg 734}}$

   (d) Determine whether the series $\sum_{k=1}^{\infty}\frac{(-1)^n\ 3n}{4n-1}$ is convergent or divergent.

   Answer: $\boxed{\text{Example 2 Sec 11.5, pg 734}}$

(e) Determine whether the series $\sum_{k=1}^{\infty}(-1)^{n+1}\dfrac{n^2}{n^3+1}$ converges or diverges.

Answer: $\boxed{\text{Example 3 Sec 11.5, pg 734}}$

11. (a) Evaluate $\lim_{n\to\infty} e^{-n}\sqrt{n}$.

Answer: Apply L'Hospital's rule once to compute $\boxed{0}$.

(b) Determine whether $\sum_{n=0}^{\infty} e^{-n}\sqrt{n}$ converges or diverges.

Answer: The series converges. Which test to use? You can use limit comparison test (compare the terms with any geometric sequence with ratio between $\dfrac{1}{e}$ and 1, for example, $b_n = \dfrac{1}{2^n}$ OR any p-sequence $b_n = \dfrac{1}{n^p}$ where $p > 1$). You can also use the ratio test because you see an exponential factor e^{-n}.

(c) Evaluate $\lim_{n\to\infty} \dfrac{(\ln(n))^2}{n^2}$.

Answer: Apply L'Hospital's rule once to compute $\boxed{0}$.

(d) Determine whether $\sum_{n=1}^{\infty} \dfrac{(\ln(n))^2}{n^2}$ converges or diverges.

Answer: The series converges.

You can use limit comparison test (compare the terms with any p-sequence $b_n = \dfrac{1}{n^p}$ where $1 < p < 2$). The Ratio Test does not work because you only see logarithmic and polynomial factors in the terms.

(e) Suppose $\sum_{n=1}^{\infty} a_n$ is a series with the property that

$$a_1 + a_2 + \cdots + a_n = 2 - 3(0.8)^n.$$

State whether $\sum_{n=1}^{\infty} a_n$ converges or diverges. If it converges, find its sum.
Answer: The expression above is the nth partial sum

$$S_n = 2 - 3(0.8)^n.$$

By definition, the series converges to

$$\lim_{n\to\infty} S_n = \boxed{2}.$$

12. Determine whether each series converges or diverges.

$a.)\ \sum_{n=1}^{\infty} \dfrac{\ln\ n}{n}$ $\quad b.)\ \sum_{n=4}^{\infty} \dfrac{1}{2^n - 9}$ $\quad i.)\ \sum_{n=1}^{\infty} \dfrac{2n^2+3n}{\sqrt{5+n^5}}$ $\quad ii.)\ \sum_{n=1}^{\infty} \dfrac{5}{2n^2+4n+3}$

Answer: `https://egunawan.github.io/spring18/notes/hw11_4_to_worksheet_key.pdf`
More practice examples of series with only positive terms: `https://egunawan.github.io/spring18/notes/notes11_strategy_pos_terms_practice.pdf`