Chains and Antichains in the Bipartite Cambrian and Tamari Lattices

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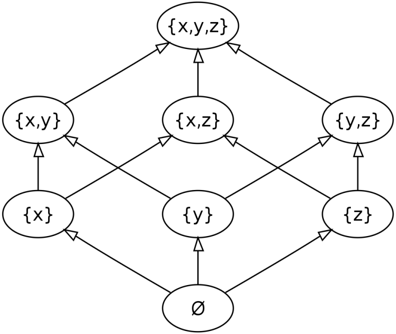
UCONN REU

POSETS



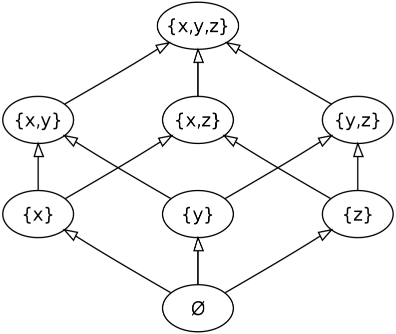
Partially Ordered Set (Poset): A set equipped with a relation $\leq$

Comparing Elements in the Poset

$$\{x, y\} \leq \{x, y, z\}$$

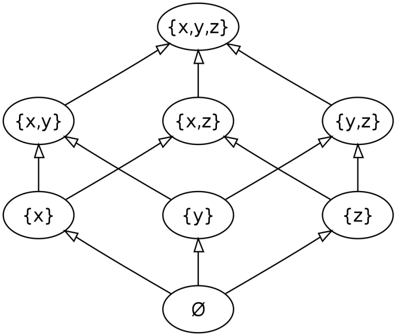
$$\{x, y\} \not\leq \{y, z\}$$

POSETS



Partially Ordered Set (Poset): A set equipped with a relation $\leq$

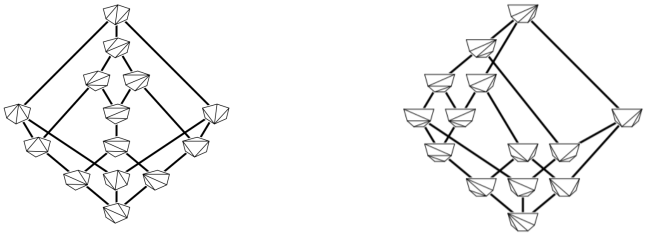
POSETS



Partially Ordered Set (Poset): A set equipped with a relation $\leq$ for which the following hold:

1. *Reflexivity*: $a \leq a$
2. *Transitivity*: if $a \leq b$ and $b \leq c$, then $a \leq c$
3. *Antisymmetry*: if $a \leq b$ and $b \leq a$, then $a = b$

Today's Talk



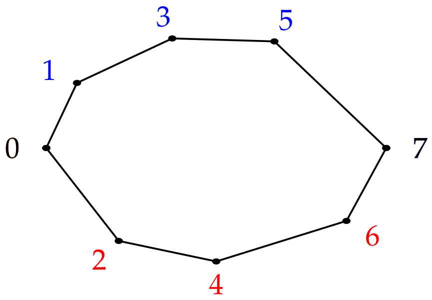
- Describe the Bipartite Cambrian Lattice & Tamari Lattice
- Prove interesting properties of these two posets

Note: A lattice is a type of poset

Part 1
The Bipartite Cambrian Lattice

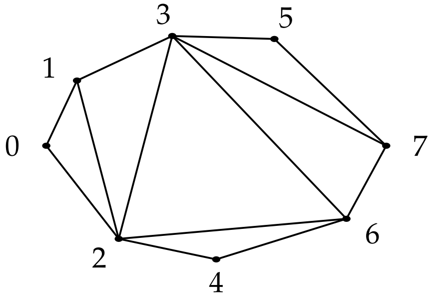
[Reading; 2012]

Bipartite polygon ($n = 6$)



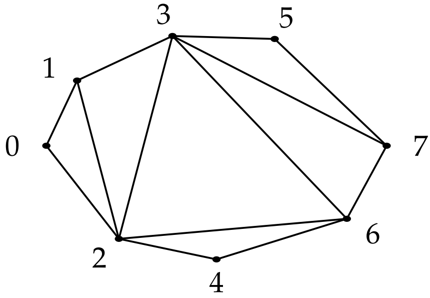
- Draw a polygon with $n + 2$ vertices
- Label vertices odd and even as above

THE ELEMENTS: TRIANGULATIONS



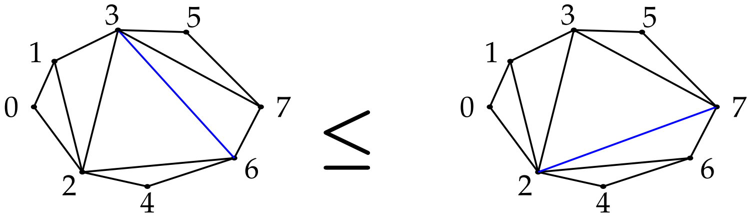
- Triangulate the polygon

THE ELEMENTS: TRIANGULATIONS



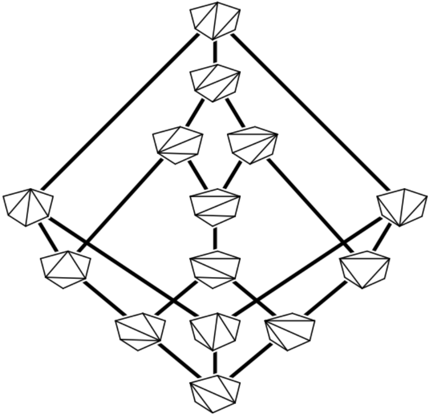
- Triangulate the polygon
- ***These triangulations are the elements in our poset!***

Defining: Comparable Elements



- Two polygons sharing an arrow differ by one **diagonal flip**
- Arrow points towards polygon with the more positive diagonal

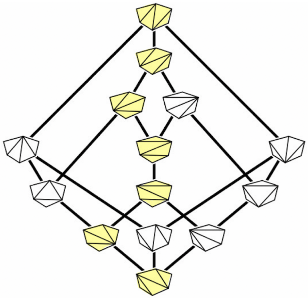
Bipartite Cambrian Lattice

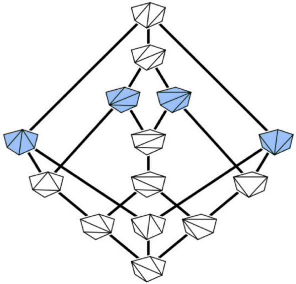


[Reading; 2012]

Note: Elements listed higher in the poset have higher-sloped diagonals

CHAINS AND ANTICHAINS

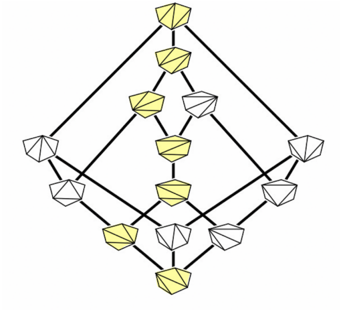




- **Chain**: A subset in which every pair of elements is comparable
- **Chain Size**: # elements in the chain

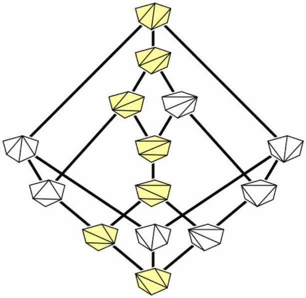
- **Antichain**: A subset in which every pair of elements is incomparable
- **Antichain Size**: # elements in the antichain

Research Problem



Question: For all n, how many maximum-length chains share only their first and last elements?

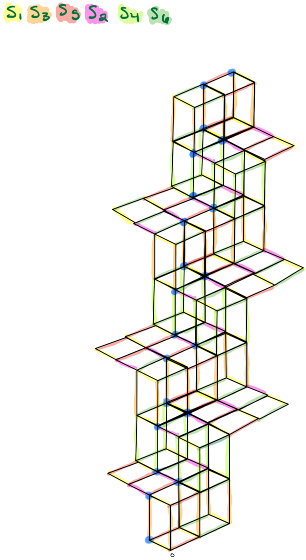
Research Problem



Question: For all n, how many maximum-length chains share only their first and last elements?

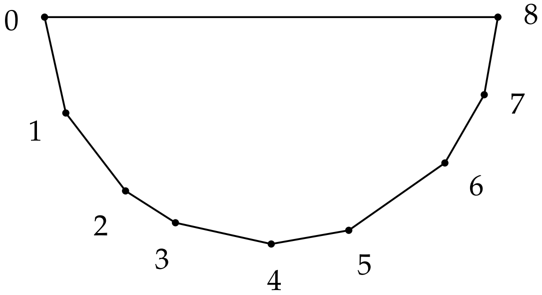
Theorem: The maximum attainable number is $\lfloor \frac{n-1}{2} \rfloor$

ONE KEY IDEA: FOCUS ON A SPECIAL SUBPOSET



Part 2
The Tamari Lattice

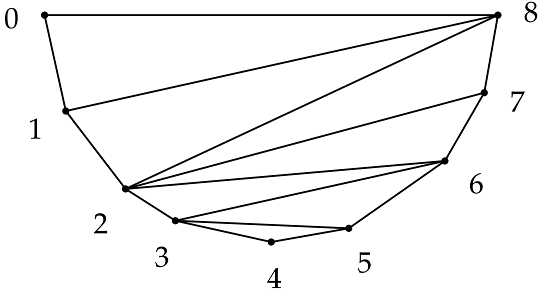
Creating the Tamari Lattice



- Draw a polygon with $n + 2$ vertices
- Label vertices in increasing order

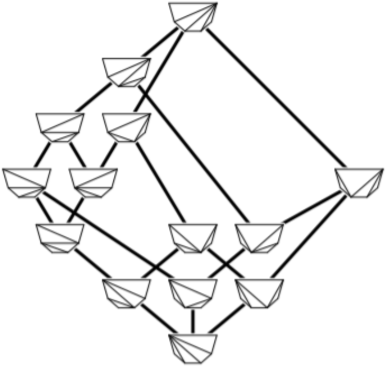
Note: The only difference from before is how we order the vertices!

Creating the Tamari Lattice



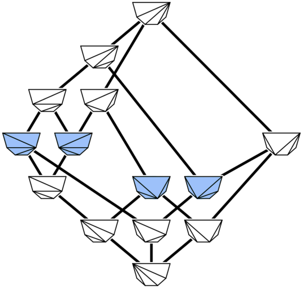
- Triangulate the polygon
- ***These triangulations are the elements in our poset!***

The Tamari Lattice



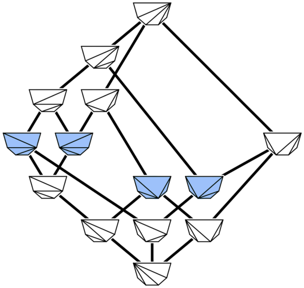
Note: As before, elements listed higher in the poset have higher-sloped diagonals

Research Results



Question: For all n, what is the size of the largest antichain?

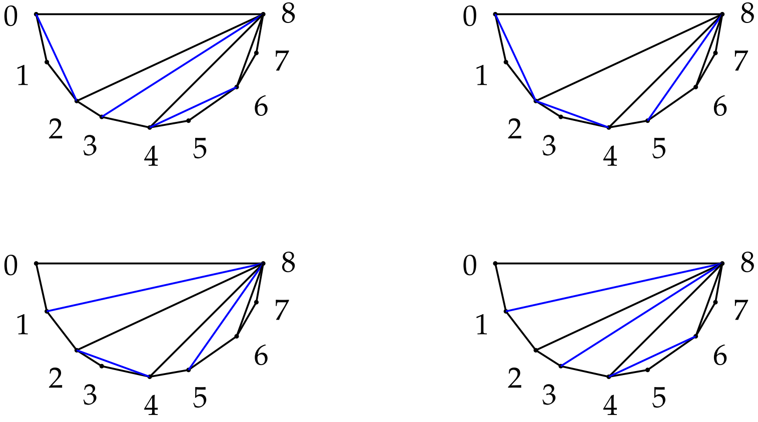
Research Results



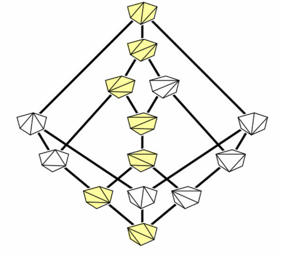
Question: For all n, what is the size of the largest antichain?
Theorem: The largest antichain has size at least

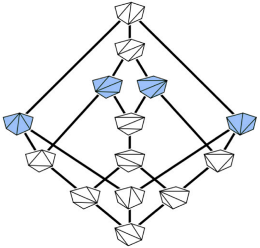
$$\binom{\lfloor \frac{n}{2} \rfloor}{\lfloor \frac{n}{4} \rfloor} \approx \frac{2^{n/2}}{\sqrt{n}}$$

Key Idea: Flip only certain "special" lines



Greene-Kleitman Theorem





Theorem [Greene, Kleitman; 1976]

- A_k = size of the largest union of k chains of P ($A_0 = 0$)
- D_k = size of the largest union of k antichains of P ($D_0 = 0$)
- $\lambda_k = A_k - A_{k-1}$ for all k, and $\lambda := (\lambda_1, \lambda_2, ...)$
- $\mu_k = D_k - D_{k-1}$ for all k, and $\mu := (\mu_1, \mu_2, ...)$

Then, λ and μ are partitions, and they are conjugate.

Our Theorems Rewritten

1. Largest union of disjoint chains in Bipartite Cambrian Lattice:

$$\lambda_1 - 2 = \lambda_2 = \lambda_3 = \cdots = \lambda_{\lfloor \frac{n-1}{2} \rfloor} > \lambda_{\lfloor \frac{n-1}{2} \rfloor + 1}$$

2. Largest antichain in Tamari Lattice:

$$\mu_1 \geq \frac{2^{n/2}}{\sqrt{n}}$$

Thank You!