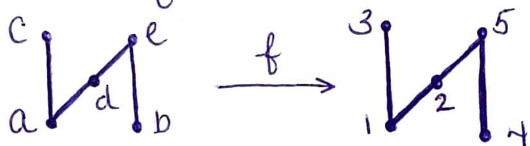
Promotion & Evacuation
by Richard Stanley

Sec 2:

remember: for a poset P with $s, t \in P$, we write $s \lessdot t$ if t covers s, that is if $s < t$ and no $u \in P$ satisfies $s < u < t$.

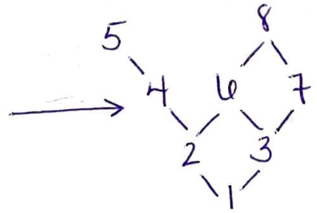
def: a linear extension of a poset P with p elements is a bijection $f: P \to [p] = \{1, 2, \dots, p\}$ such that if $s < t$ for $s, t \in P$, then $f(s) < f(t)$

ex:



ex: Young Tableaux are linear extensions of finite order ideals in the infinite poset $\mathbb{N} \times \mathbb{N}$ (wiki)

1	2	4	5
3	6		
7	8		

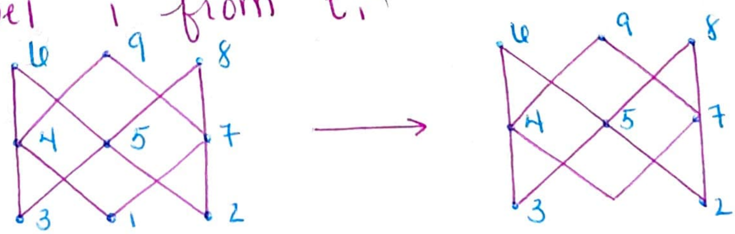


def: we define Promotion as a bijection $\partial: \mathcal{L}(P) \to \mathcal{L}(P)$ as follows:

Note: think of each element $t \in P$ as having a label $f(t)$

① consider $t_1 \in P$ such that $f(t_1) = 1$. Remove the label 1 from t_1

ex



some linear extension of P

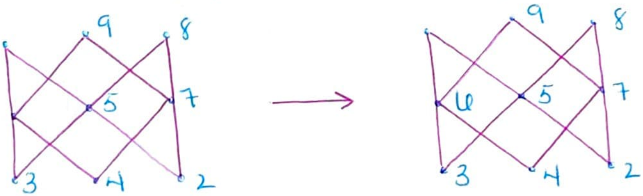
② consider the elements covering t_1
Let t_2 be the one with the smallest label $f(t_2)$.
Remove this label at t_2 and place it at t_1

ex

- imagine $f(t_2)$ "sliding" to t_1

③ consider the elements covering t_2
Let t_3 be the one with the smallest label $f(t_3)$.
Remove this label at t_3 and place it at t_2

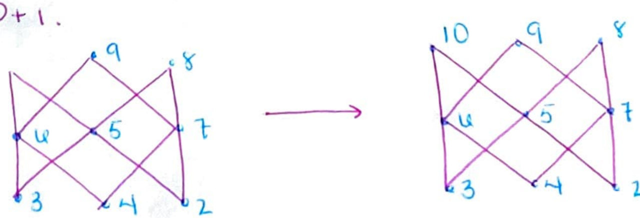
ex



④ Repeat the process of steps 2 and 3 until reaching a maximal element t_k of P.

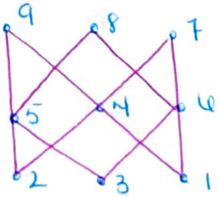
⑤ after "sliding" $f(t_k)$ down to t_{k-1}, label t_k with $p+1$.

ex



⑥ subtract 1 from every label.

$fd =$



$\in \mathcal{L}(P)$

- we now have a new linear extension $f\partial \in \mathcal{L}(P)$
- Note: $t_1 <\cdot t_2 <\cdot t_3 <\cdot \ldots <\cdot t_k$ is a maximal chain of P
  - $\Rightarrow$ we call this chain the <u>promotion chain</u> of f

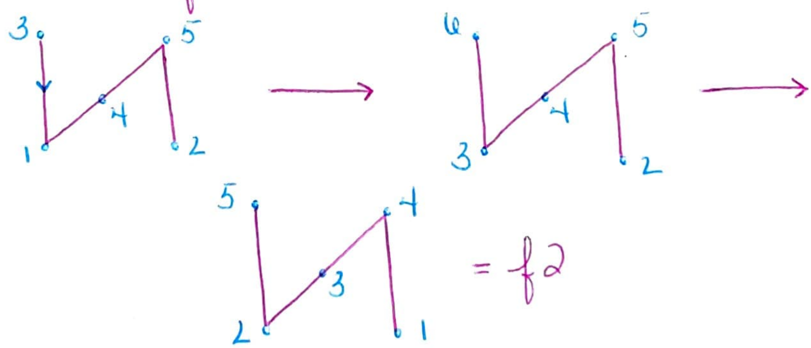
<u>def</u>: Dual Promotion, ∂^*, works in reverse to ∂.
- remove the largest $f(u_1)$ from $u_1 \in P$
- Consider the elements that u_1 covers.
- Slide the largest of these labels up to u_1

  ⋮ (continue process)
- after reaching a minimal element u_k, label it 0
- add 1 to each label, obtaining $f\partial^*$

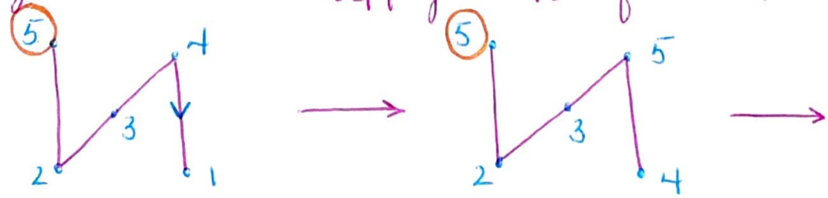
- Note: it is known that $\boxed{\partial^{-1} = \partial^*}$

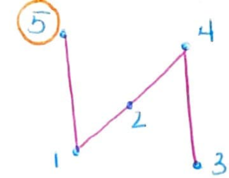
<u>def</u>: <u>Evacuation</u> is a variant of Promotion
- it takes $f \in \mathcal{L}(P)$ to produce another linear extension of P denoted $f\varepsilon$

① apply ∂ to f.

ex 

② "freeze" label p. apply ∂ to $f\partial - \{p\}$

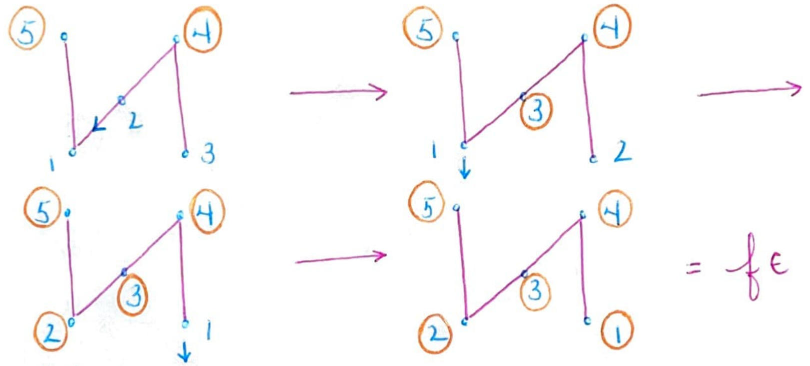
ex 



$= (f\partial - \{p\})\partial$

③ freeze label $p-1$. apply ∂ to $f\partial - \{p, p-1\}$

⋮ (continue process until every element is frozen)



$= f\varepsilon$

- Note: we define dual evaluation in a similar way to dual promotion
  $\Rightarrow$ Denote it ε^*.

Results

Theorem 2.1 (Schützenberger)

Let P be a p-element poset.
The operators ε, ε^*, and ∂ satisfy the following:

(a) evaluation is an involution
$\Rightarrow \varepsilon^2 = 1$ (the identity)

(b) $\partial^p = \varepsilon\varepsilon^*$

(c) $\partial\varepsilon = \varepsilon\partial^{-1}$

- The proof of Theorem 2.1 requires some background information regarding Lemma 2.2

Lemma 2.2

Let G be a group with generators $T_1, T_2, \ldots, T_{p-1}$ such that:

$$T_i^2 = 1, \quad 1 \leq i \leq p-1$$
$$T_iT_j = T_jT_i, \text{ if } |i-j| > 1$$

→ notice that G is actually a Coxeter group!

Now, define the following:

$$\delta = T_1 T_2 \ldots T_{p-1}$$
$$\gamma = (T_1T_2 \cdots T_{p-1})(T_1T_2 \cdots T_{p-2}) \cdots (T_1T_2)(T_1)$$
$$\gamma^* = (T_{p-1}T_{p-2} \cdots T_1)(T_{p-1}T_{p-2} \cdots T_2) \cdots (T_{p-1}T_{p-2})(T_{p-1})$$

- Lemma 2.2 states...

In the group G, we have the following identities:

(a) $\gamma^2 = (\gamma^*)^2 = 1$

(b) $\delta^p = \gamma\gamma^*$

(c) $\delta\gamma = \gamma\delta^{-1}$

- Proofs for parts a, b, and c are quite similar and make use of induction.

Proof of Lemma 2.2 (a)

Base Case: Let $p = 2$.

$\gamma_2 = T_1 \longrightarrow \gamma_2^2 = T_1 \cdot T_1 = 1$ ✓

Inductive Step: assume for $p-1$. Consider γ_p^2.

$$\gamma_p^2 = (T_1T_2 \cdots T_{p-1})(T_1T_2 \cdots T_{p-2}) \cdots (T_1T_2)(T_1) \cdot (T_1T_2 \cdots T_{p-1})(T_1T_2 \cdots T_{p-2}) \cdots (T_1T_2)(T_1)$$

Notice that we can cancel the two middle T_1's. Now we can cancel the T_2's.

$\rightarrow \gamma_p^2 = (T_1T_2 \cdots T_{p-1})(T_1T_2 \cdots T_{p-2}) \cdots (T_1T_2T_3)(T_1)$
$\cdot (T_3T_4 \cdots T_{p-1})(T_1T_2 \cdots T_{p-2}) \cdots (T_1T_2)(T_1)$

- Ah ha! But now we can swap T_3 with T_1 to cancel the T_3's.

$\rightarrow \gamma_p^2 = (T_1T_2 \cdots T_{p-1})(T_1T_2 \cdots T_{p-2}) \cdots (T_1T_2T_1)(\cancel{T_3}$
$\cdot (\cancel{T_3}T_4 \cdots T_{p-1})(T_1T_2 \cdots T_{p-2}) \cdots (T_1T_2)(T_1)$

- If we continue this process of swapping T_i with T_j and canceling like terms, eventually we end up with γ_{p-1}^2 which equals 1 by the base case.
- Then by induction, $\gamma_p^2 = 1$ ■

example of Proof:

Let's use γ_3^2 and γ_4^2.

$$\gamma_3^2 = (T_1T_2T_3)(T_1T_2)(T_1) \cdot (T_1T_2T_3)(T_1T_2)(T_1)$$

$$\gamma_4^2 = (T_1T_2T_3T_4)(T_1T_2T_3)(T_1T_2)(T_1) \cdot (T_1T_2T_3T_4)(T_1T_2T_3)(T_1T_2)(T_1)$$

Goal: Show that at some point, $\gamma_4^2 = \gamma_3^2$ in the cancellation process.

First, lets convince ourselves that $\gamma_3^2 = 1$:

$$\gamma_3^2 = T_1T_2T_3\underline{T_1}\cancel{T_2}\cancel{T_1}T_1\cancel{T_2}\underline{T_3}T_1T_2T_1$$
$$= T_1T_2\cancel{T_3}\cancel{T_3}\cancel{T_1}T_1T_2T_1$$
$$= \cancel{T_1}\cancel{T_2}T_2\cancel{T_1} = 1 \checkmark$$

We show $\gamma_4^2 = \gamma_3^2$ by canceling the following elements:

$$\Big[\gamma_4^2 = (T_1 T_2 T_3 \cancel{T_4})(T_1 T_2 \cancel{T_3})(T_1 \cancel{T_2})(\cancel{T_1}) \cdot (\cancel{T_1} \cancel{T_2} \cancel{T_3} \cancel{T_4})(T_1 T_2 T_3)(T_1 T_2)(T_1)\Big]$$

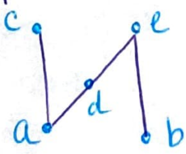
$$\begin{aligned}
\gamma_4^2 &= (T_1 T_2 T_3 T_4)(T_1 T_2 T_3)(T_1 \cancel{T_2})(\cancel{T_1})(\cancel{T_1} \cancel{T_2} T_3 T_4)\cdots \\
&= (T_1 T_2 T_3 T_4)(T_1 T_2 \cancel{T_3} \cancel{T_3})(T_1 T_4)\cdots \\
&= (T_1 T_2 T_3)(T_1 T_4 T_2)(T_1 T_4)\cdots \\
&= (T_1 T_2 T_3)(T_1 T_2)(T_4 T_1 T_4)\cdots \\
&= (T_1 T_2 T_3)(T_1 T_2)(T_1)(\cancel{T_4} \cancel{T_4})\cdots \\
&= \gamma_3^2 = 1
\end{aligned}$$

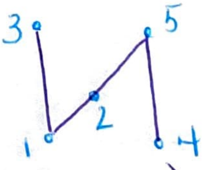
Proof outline of Theorem 2.1

- Notice that Lemma 2.2 and Theorem 2.1 are very similar and, in fact, they are connected!

Regard $f \in \mathcal{L}(P)$ as the word $f^{-1}(1), \ldots, f^{-1}(p)$.

For example:

If $P =$ 

and $f =$ 

then as a word (or permutation) we have $f = adcbe$

Define $T_i : \mathcal{L}(P) \to \mathcal{L}(P)$ for $1 \le i \le p-1$ to be

$$T_i(u_1 u_2 \ldots u_p) = \begin{cases} u_1 u_2 \ldots u_p & \text{if } i \text{ and } i+1 \text{ are comparable in } P \\ u_1 u_2 \ldots u_{i+1} u_i \ldots u_p & \text{if not} \end{cases}$$

By construction, $\varepsilon = \gamma$

remember: $\gamma = (T_1 T_2 \cdots T_{p-1})(T_1 T_2 \cdots T_{p-2}) \cdots (T_1 T_2)(T_1)$

Furthermore, $\varepsilon^* = \gamma^*$

remember: $\gamma^* = (T_{p-1} T_{p-2} \cdots T_1)(T_{p-1} T_{p-2} \cdots T_2) \cdots (T_{p-1} T_{p-2})(T_{p-1})$

Theorem 2.1 Computes the proof by showing $\partial = \delta$

remember: $\delta = T_1 T_2 \cdots T_{p-1}$

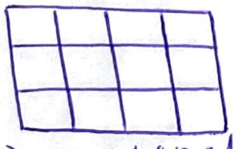
We have a proof for Lemma 2.2 so doing so completes the proof of Theorem 2.1! ∎

Special Cases of (b)

- There exist a few "nontrivial" classes of posets P in which (b) $\partial^P = \varepsilon\varepsilon^*$ is explicit

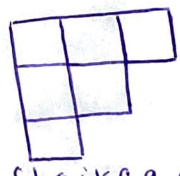
Theorem 4.1

- Remember that Standard Young Tableaux of shape λ correspond to linear extensions of a certain poset P_λ
- For the following shapes and shifted shapes with a total of p squares, we have the following:



(a) rectangle

$f\partial^p = f$



(b) staircase

$f\partial^p = f^t$ (transpose)