(cou(nt)in)g ((cha)in)s (in (the)) ((Tam)ari) ((Latt)ice)

Presentation Outline

- Discuss Binary Bracketing + its relationship to triangulations
- Introduce four Theorems on the Greene-Kleitman Invariant for the Tamari Lattice
- Proof outline for all theorems

BINARY BRACKETING

Def Parenthesized string of x's (built up of binary operations)

Ex

$$\left.\begin{array}{l} ((x\,x)x)x \\ (x(x\,x))x \\ (x\,x)(x\,x) \\ x((x\,x)x) \\ x(x(x\,x)) \end{array}\right\}\ \text{5 possible binary bracketing}$$

We can also write it as such:
(Early)

$$(((x_0\ x_1)\ x_2)\ x_3) \leftarrow \text{minimum elt.}$$
$$((x_0\ (x_1\ x_2))\ x_3)$$
$$((x_0\ x_1)\ (x_2\ x_3))$$
$$(x_0((x_1\ x_2)\ x_3))$$
$$(x_0(x_1\ (x_2\ x_3))) \leftarrow \text{maximum elt.}$$

$$\cancel{(x(x)x)x}$$

n-tuples

$$(x_0 (x_1 (x_2 \ \ x_3))) \leftarrow \begin{array}{cc} \begin{pmatrix} 1 & 2 & 3 \\ 3 & 3 & 3 \end{pmatrix} & \begin{array}{l} i\text{'s} \\ v_i\text{'s} \end{array} \end{array}$$

$$= (3, 3, 3)$$

$$(((x_0 \ \ x_1) \ x_2) \ x_3) \leftarrow \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$$

$$= (1, 2, 3)$$

Example:

$$(x_0 ((x_1 \ \ x_2) \ x_3)) \qquad \begin{pmatrix} 1 & 2 & 3 \\ 2 & 3 & 3 \end{pmatrix}$$

$$(2, 3, 3)$$

Rules

$x_0) x_1 (x_2 \, x_3$

1) If $1 \le i \le n$, then $i \le v_i \le n$

Yes	No
• $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 3 & 3 & 4 \end{pmatrix} = (1,3,3,4)$	• $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 1 & 1 & 1 & 1 \end{pmatrix} = (1,1,1,1)$
• $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 2 & 4 & 4 \end{pmatrix} = (2,2,4,4)$	• $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 1 & 2 & 4 \end{pmatrix} = (4,1,2,4)$
• $\begin{pmatrix} 1 & 2 & 3 & 4 \\ 4 & 4 & 4 & 4 \end{pmatrix} = (4,4,4,4)$	

2) If $v_i > i$, $i \le k \le v_i$, then $v_k \le v_i$

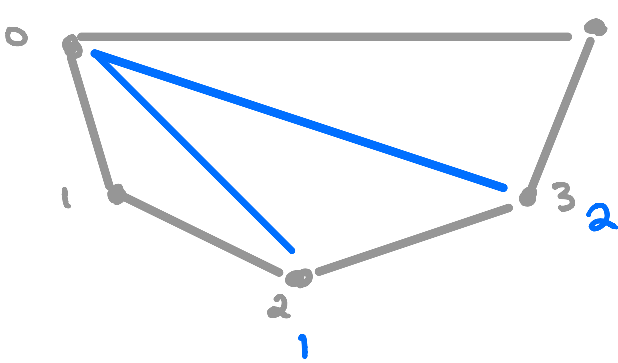
$$\begin{pmatrix} 1 & 2 & 3 & 4 \\ 3 & 3 & & \end{pmatrix} \quad \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 4 & 2 & 3 & 4 & 5 \end{pmatrix}$$

3) $(v_1, v_2, \dots v_n) \geq (u_1, u_2, \dots, u_n)$
iff $v_i \geq u_i \;\; \forall i$

$(1, 3, 3, 4, 5) > (1, 2, 3, 4, 5)$

$(2, 2, 3) \quad (1, 3, 3)$

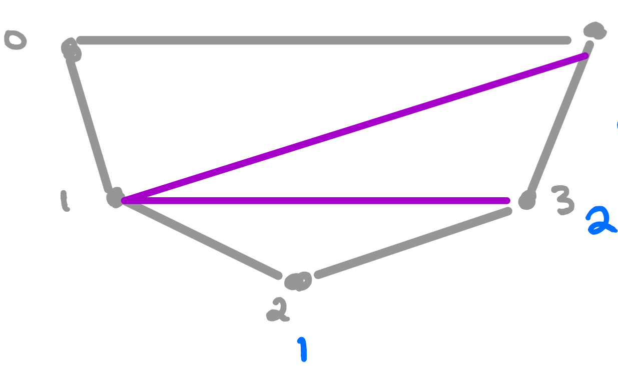
Relationship to Triangulations



$\rightarrow (1, 2, 3)$

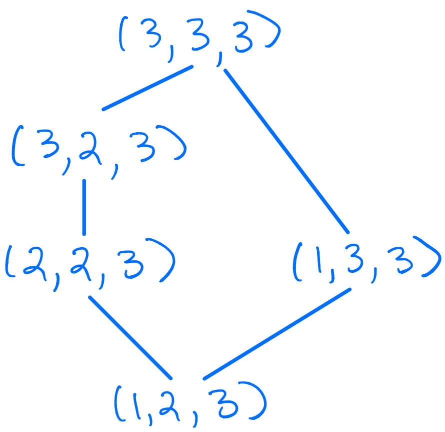
$$\begin{array}{r} i\text{'s} \\ v_i\text{'s} \end{array} \begin{pmatrix} 1 & 2 & 3 \\ 1 & 2 & 3 \end{pmatrix}$$

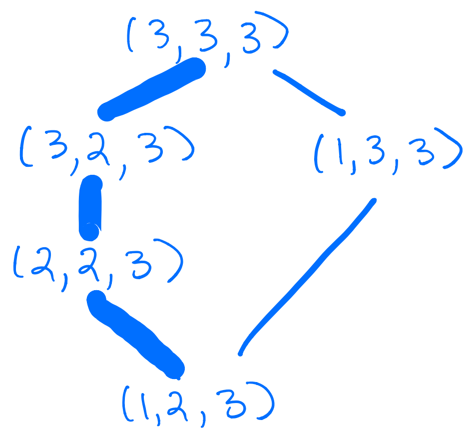
$= (1, 2, 3)$



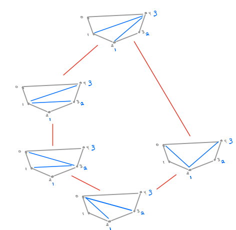
$$\rightarrow \begin{pmatrix} 1, & 2, & 3 \\ 3 & 2 & 3 \end{pmatrix}$$

$= (3, 2, 3)$





Tamari Lattice n=3



Advantages to ~~Binary Bracketing~~ n-tuples

- Easier to analyze cover relations / comparable elements using n-tuples
- Easier to analyze lattice with...

Early Theorems

$(n \geq 4)$

A) For $n > 5$, $\lambda_2(T_n) = \lambda_1(T_n) - 4$

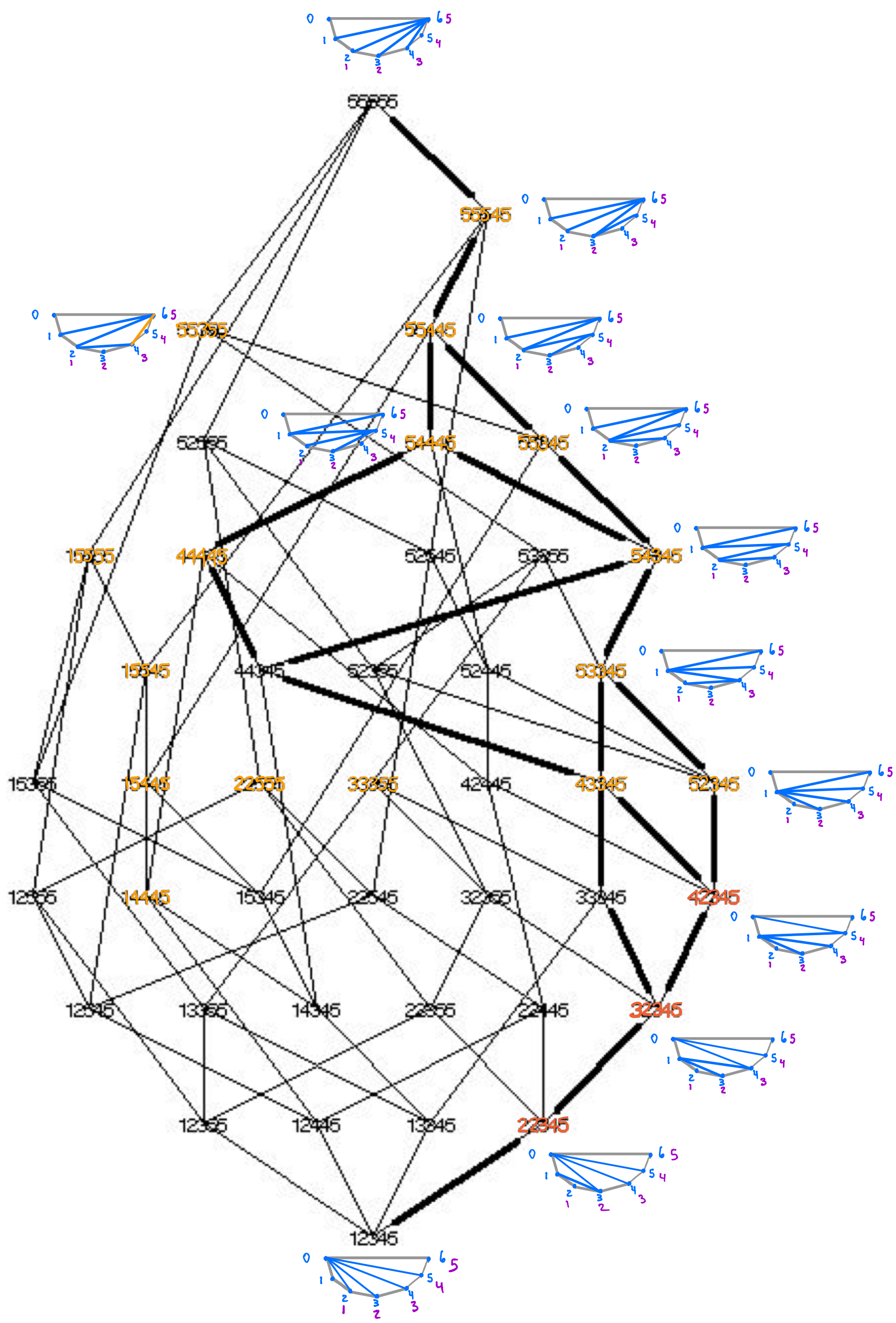
B) For $n > 6$, $\lambda_3(T_n) = \lambda_2(T_n) - 2$

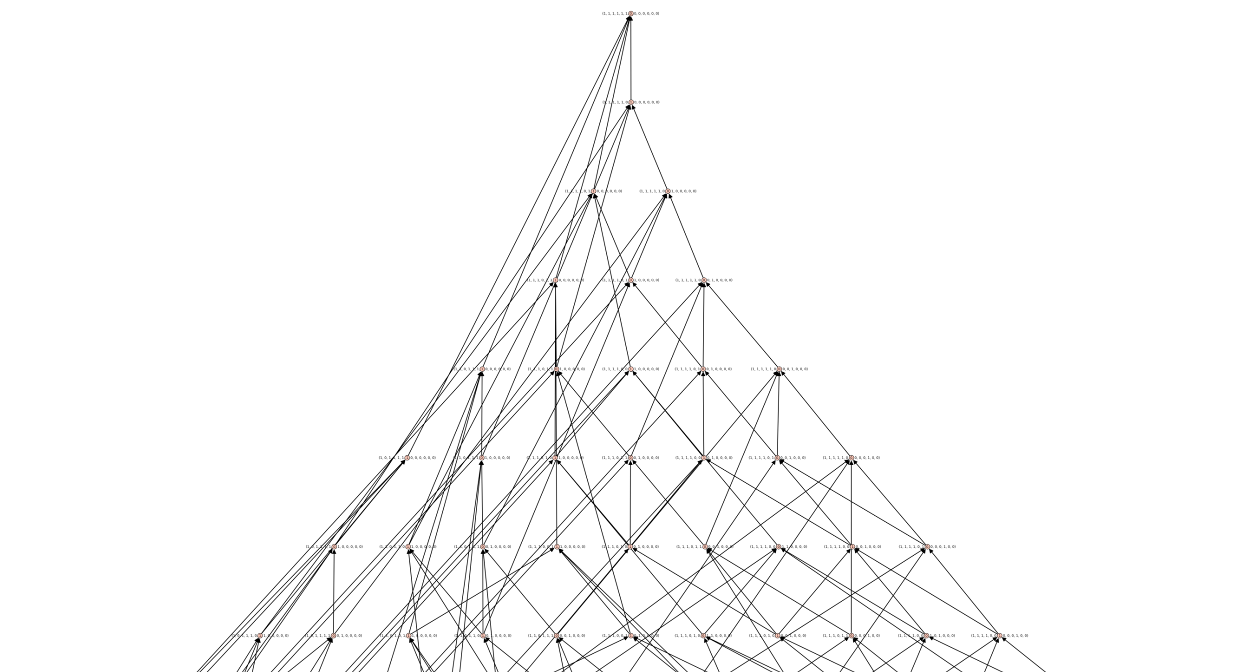
Outline

A:
- Find largest union of two disjoint chains
- Show you can't do any better than that (i.e. find an upper bound)

B:
- Find largest union of three disjoint chains
- " " " " " "

Key Concepts:





One more definition....

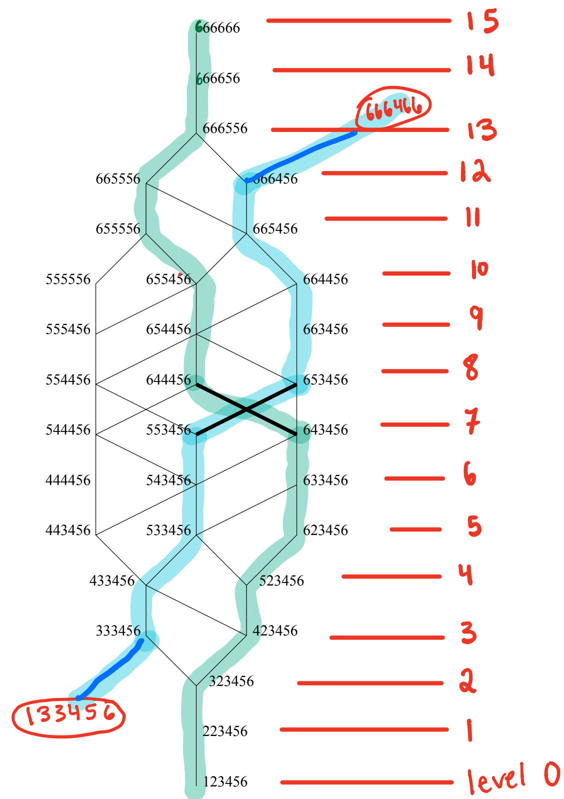
<u>Def</u> U_n = Subset of T_n containing n-tuples with no jumps
$(v_{i+1} - v_i < 2)$

In U_n	Not In U_n
(1, 2, 3, 4)	(1, 3, 3, 4)
(2, 2, 3, 4)	(1, 2, 4, 4)
(4, 2, 3, 4)	(4, 2, 4, 4)

Construction

1) $2, n-1, n)$, and then ends at $(n, n, \ldots, n, n-2, n, n)$. This chain has four fewer elements than the long one, as desired, and it is not hard to see that 2) the two chains are disjoint. The key idea here was to take advantage of the crossing covers highlighted in Figure 2. Note that it would have been easier to construct two chains of length $\frac{n(n-1)-2}{2}$ by not having them cross, but it is interesting to see that we really can get disjoint chains of length $\lambda_1(T_n)$ and $\lambda_2(T_n)$.

T_6



$\Sigma n = \frac{n(n+1)}{2}$

Figure 2: The poset U_6.

To show that this does indeed give us $\lambda_2(T_n) = \lambda_1(T_n) - 4$, we prove that T_n can be decomposed into $N = \lambda_1(T_n) = \frac{n(n-1)}{2} + 1$ antichains, four of which consist of a single element. To this end, we start with the most obvious decomposition into N antichains.

16 anti-chains

Draw the Hasse diagram of T_n by starting with $(1, 2, \ldots, n)$ at the bottom, as level 0. Now each subsequent level consists of the minimal elements of what's left of T_n. It is not hard to see that level i will consist of the elements of T_n whose components sum to $\frac{n(n+1)}{2} + i$. These levels give us a decomposition of T_n into N antichains, but unfortunately only the top two levels ($(n, n, \ldots, n)$ and $(n, n, \ldots, n, n-1, n)$) and the bottom level have

Construction

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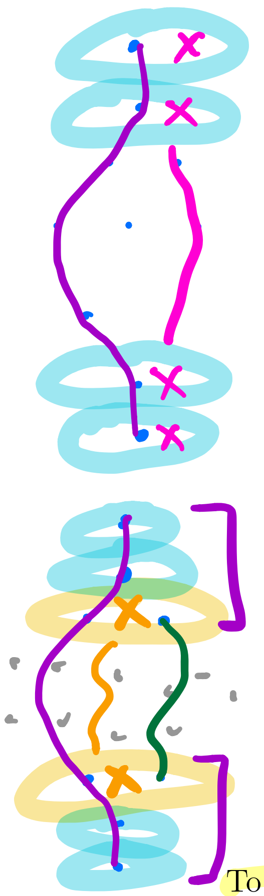

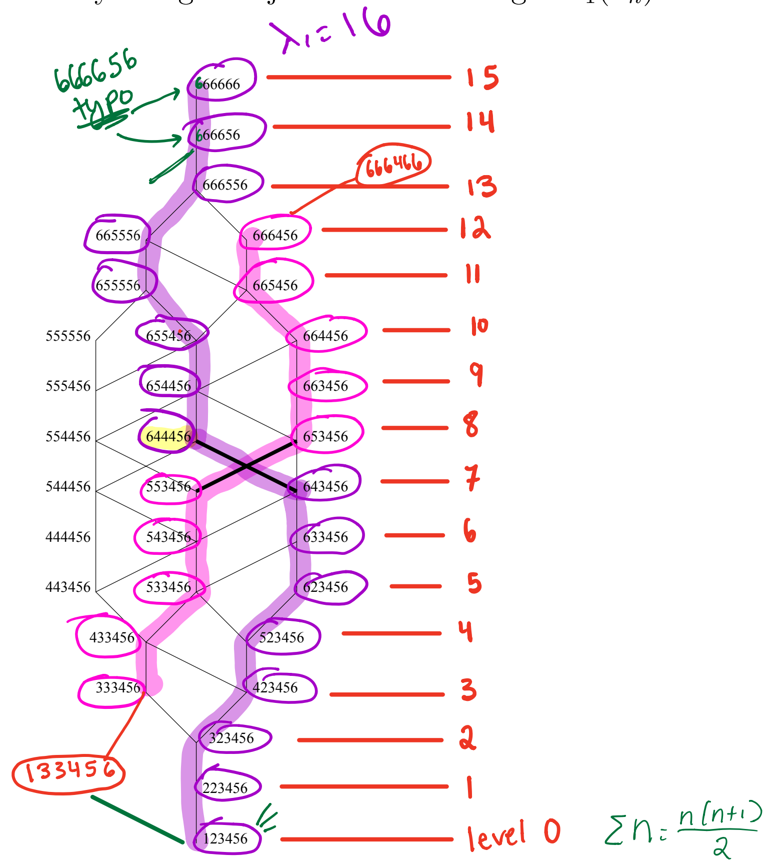

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Lim & Zhang Theorems

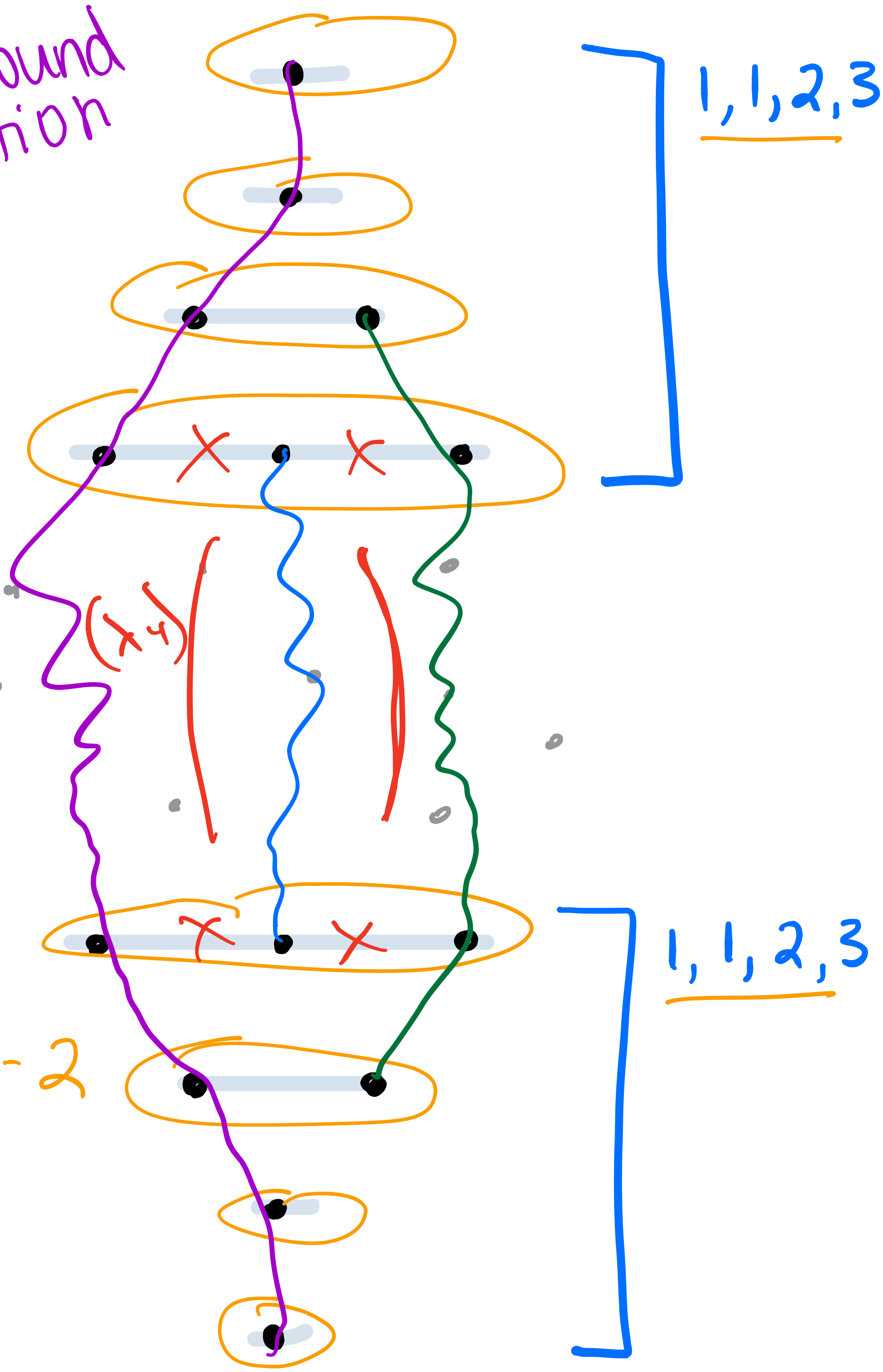
- For sufficiently large n,

  $\lambda_4(T_n) = \lambda_3(T_n) - 2, \quad \lambda_5(T_n) = \lambda_4(T_n)$

Outline

- Give an elegant solution to an upper bound
- Construct chains with brute force

Upper Bound Construction



$\lambda_4 = \lambda_3 - 2$

$\lambda_5 = \lambda_4$