# Presentation on Britz and Fomin

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# Introduction -

Notation I will be using:

- $\lambda(P)$ is the partition of the poset $P$ given by the usual "union of chains and antichains."
- $|P|$ denotes the number of elements in the poset $P$ (= $n$ typically)
- $P$ denotes a poset unless otherwise stated.

# Main Important Theorems -

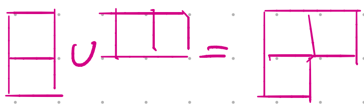
(Def) Subsets of partitions.

A partition $\alpha \subset \beta$ if $\alpha$ is a partition and "sits inside" the young diagram of $\beta$.

Ex|

Rmk: Unions and subtractions work similarly.

Ex.

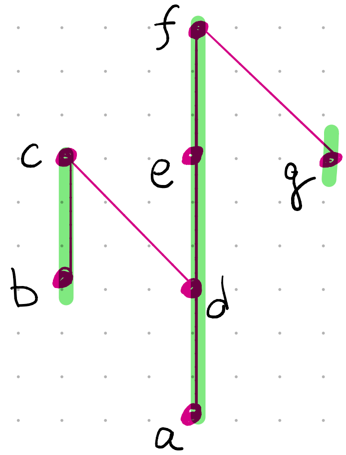


## The "Monotonicity Theorem"

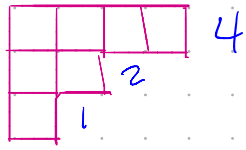
STATEMENT: Let $p \in P$ and $P' = P - \{p\}$. If $p$ is minimal or maximal, then $\lambda(P') \subset \lambda(P)$ as partitions.

## Ex]

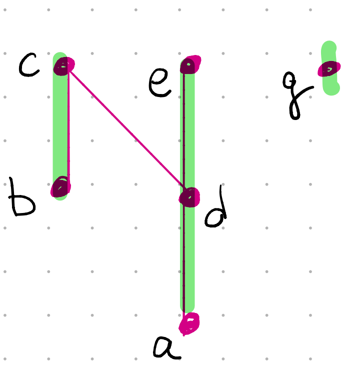
Let $P$ =



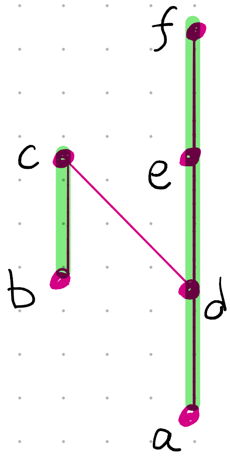

Note: $\lambda(P) =$



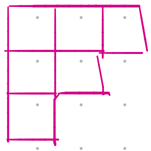

Let $P'$ =



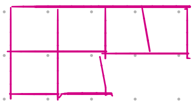

Let $P''$ =




Note: $\lambda(P') =$



Note: $\lambda(P'') =$



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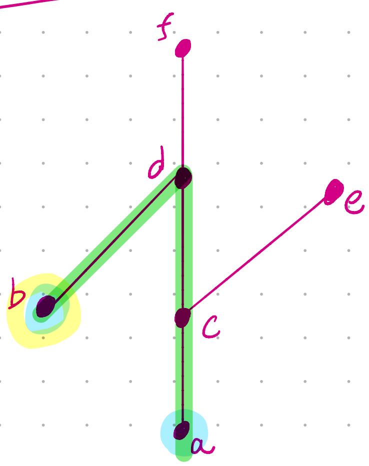
**Def (Order Ideals)** | (This concept is sort of analogous to ring-theoretic ideals) An order ideal $I$ is a subset of a poset $P$, such that if a given element $x$ of $P$ is in $I$, then all elements less than $x$ are also in $I$.

## Ex]

Example Order ideals highlighted in this Hasse diagram:




$\{a, b\}$

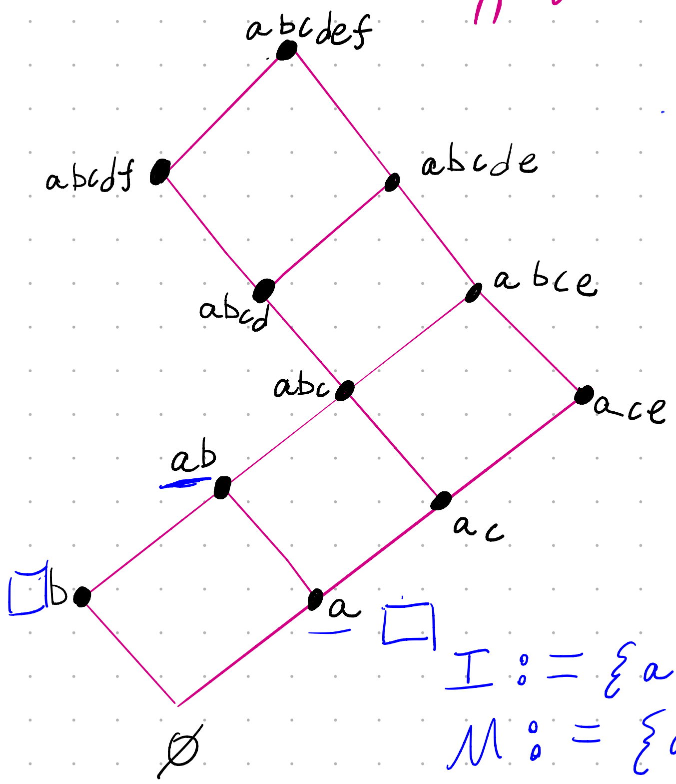
$\{b\}$

$\{a, b, c, d\}$

NOT order ideals (for example)

$\{a, c, d\}$, $\{e, d\}$, $\{b, e\}$

List of all order ideals ordered by inclusion:
(set braces dropped for convenience)




$I := \{a\}$
$M := \{a\}$

## The Recursive Construction Theorem

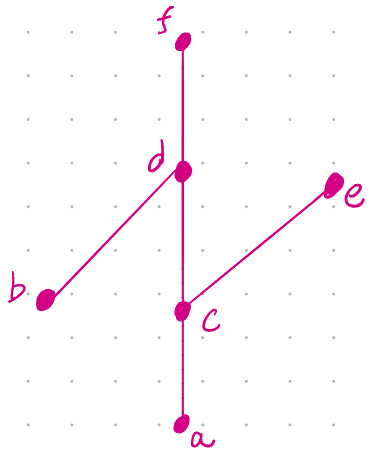
As it turns out, we can use order ideals to recursively build up $\lambda(P)$.

$I := \{a, b\}$
$M := \{a, b\}$

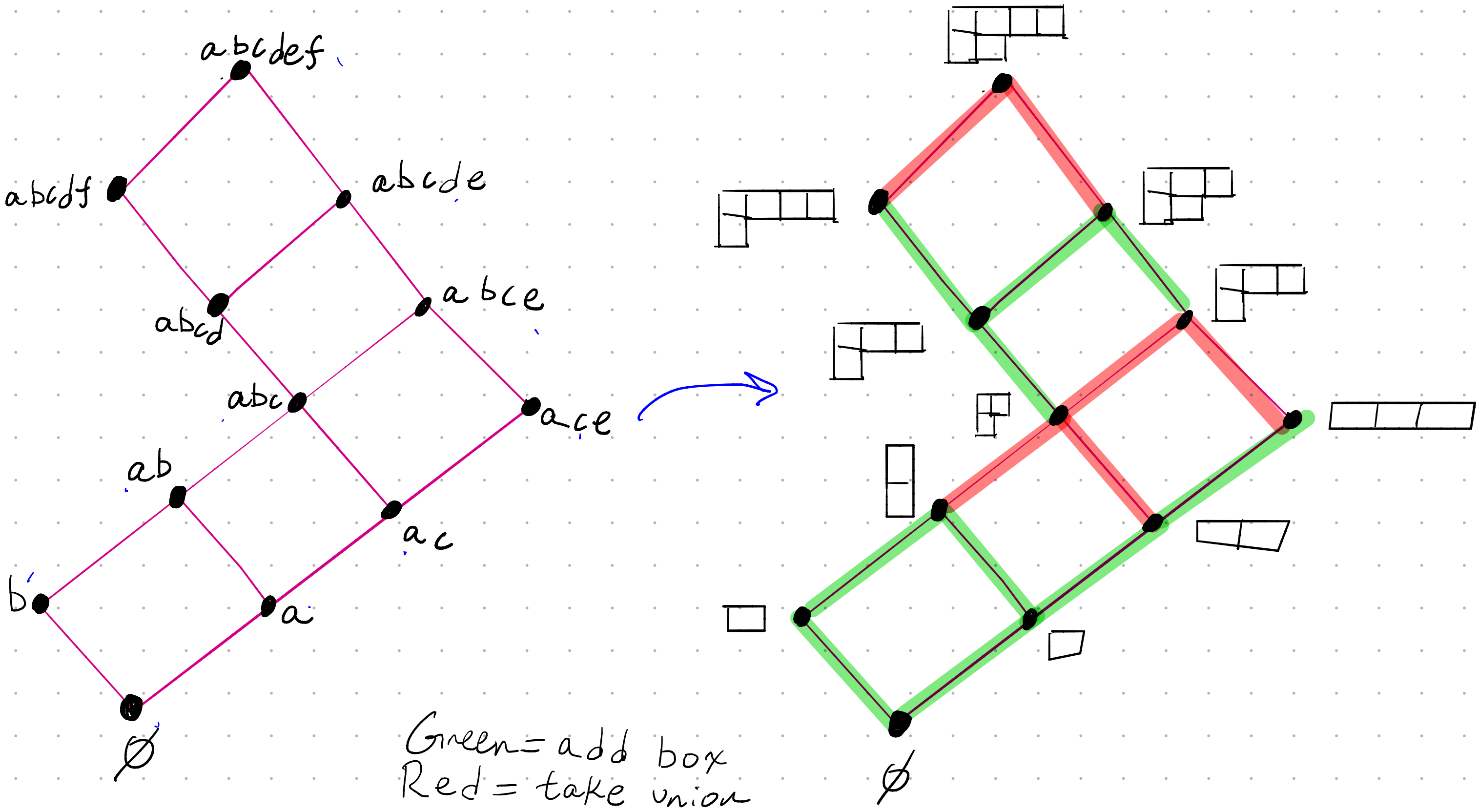
## ALGORITHM

- SET $I :=$ a smallest nonempty order ideal of $P$
- FOR every order ideal of $P$ DO:
  - DETERMINE the list $M = \{p_1, \dots p_k\}$ of maximal elements of $I$
  - IF $\lambda(I - \{p_1\}) = \dots = \lambda(I - \{p_k\})$ DO:
    - SET $\lambda' = \lambda(I - \{p_1\})$
    - RESULT:
      - $\underline{\lambda(I)} = \overset{\emptyset}{\lambda'}$ + 1 box at the end of the $k^{th}$ row of $\lambda'$
      - $\lambda(\{a\})$
  - ELSE # (i.e. $\lambda(I - \{p_i\}) \neq \lambda(I - p_j)$) DO:
    - RESULT: $\lambda(I) = \bigcup_i (I - \{p_i\})$
    - # (Follows from the monotonicity theorem)
  - Set $I :=$ the next order ideal up / on the same level

## Ex 1




$\mathcal{L}(P)$




Green = add box
Red = take union

## ALGORITHM (From before)

- SET $I :=$ the smallest order ideal of $P$
- FOR every order ideal of $P$ DO:
  - DETERMINE the list $M = \{p_1, \dots, p_k\}$ of maximal elements of $I$
  - IF $\lambda(I - \{p_1\}) = \dots = \lambda(I - \{p_k\})$ DO:
    - SET $\lambda' = \lambda(I - \{p_1\})$
    - RESULT: $\lambda(I) = \lambda' + 1$ box at the end of the $k^{th}$ row of $\lambda'$
  - ELSE # (i.e. $\lambda(I - \{p_i\}) \neq \lambda(I - p_j)$) DO:
    - RESULT: $\lambda(I) = \bigcup_i (I - \{p_i\})$
    - # (Follows from the monotonicity theorem)
  - Set $I :=$ the next order ideal up/on the same level

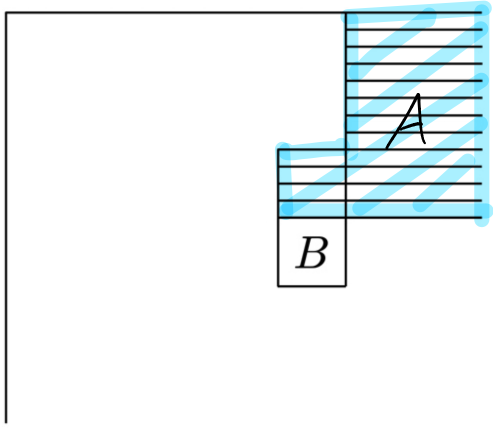
# Theorem about restrictions on the growth of $\lambda(P)$ (GRT)

Let $p_1$ and $p_2$ each be either maximal or minimal elements of a poset $P$.

Let $B$ be the box of $\lambda = \lambda(P)$ removed when $p_2$ is removed from $\lambda$. Let $A$ be the box of $\lambda$ removed when $p_1$ is removed from $\lambda - \{B\}$.

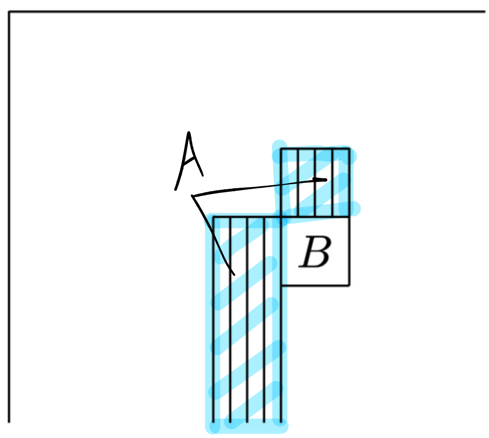
(i)

$\boxed{\text{If } p_1 \text{ and } p_2 \text{ are both maximal or both minimal}}$



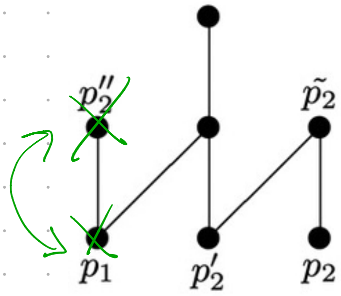

(ii)

$\boxed{\text{If } p_1 \text{ is maximal and } p_2 \text{ is minimal or vice versa}}$

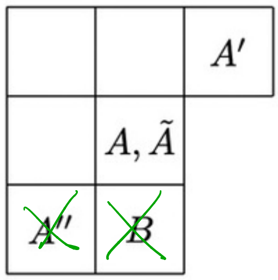



Possible locations for $A$, given $B$

Ex. Let $p_1 = p_1$ and $p_2 = p_2$. Let $A$ and $B$ be defined as above. Then let $p_2 = p_2', p_2'', \tilde{p}_2$, and $A = A', A'', \tilde{A}$ resp.




(a) $P$

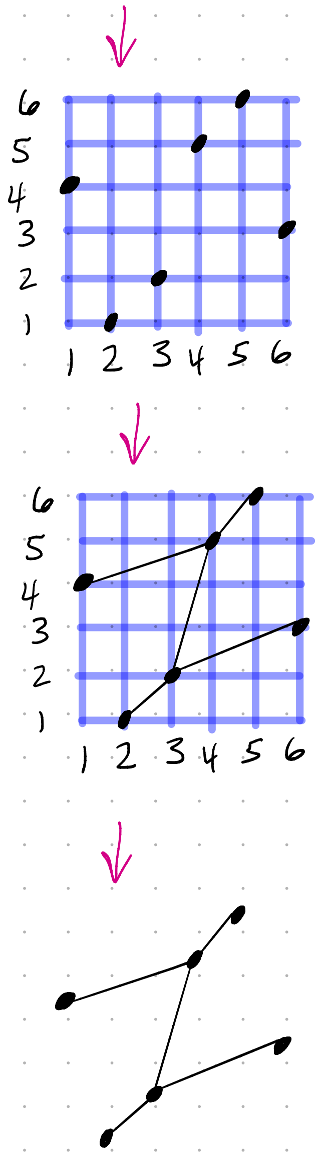



(b) $\lambda(P)$

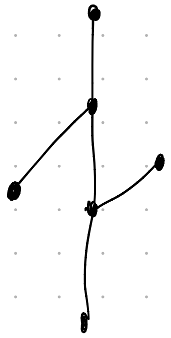
# Permutation Posets

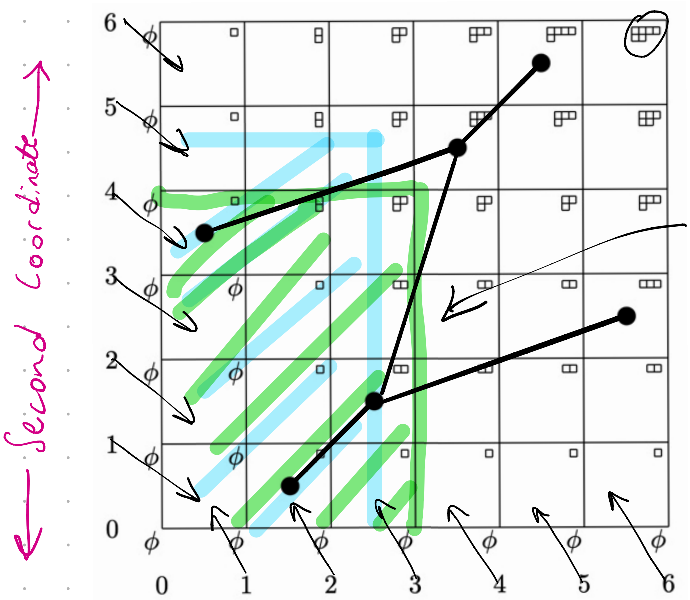
Recall: A "perm. poset" can be formed by the following procedure:

Ex| Let $\pi = 412563$



# Growth Diagrams





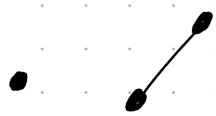

$P(3,5)$

Poset $P$

FIGURE 12. The growth diagram for $\sigma = 412563$

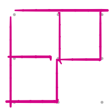
First Coordinate

Denote $P(i,j)$ to be the poset formed by taking the sub-poset of $P$ defined by the points of $P$ lying in cells weakly southwest ($\swarrow$) of the cell $(i,j)$

Ex | $P(3,5)$ is 

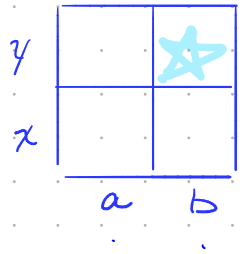
Denote $\lambda_{ij} := \lambda(P(i,j))$

Ex |

$\lambda_{35} =$ 

# Local Theorem Regarding Growth Diagrams

Consider a 2x2 sub-growth diagram




: $b = 1+a$, $y = 1+x$

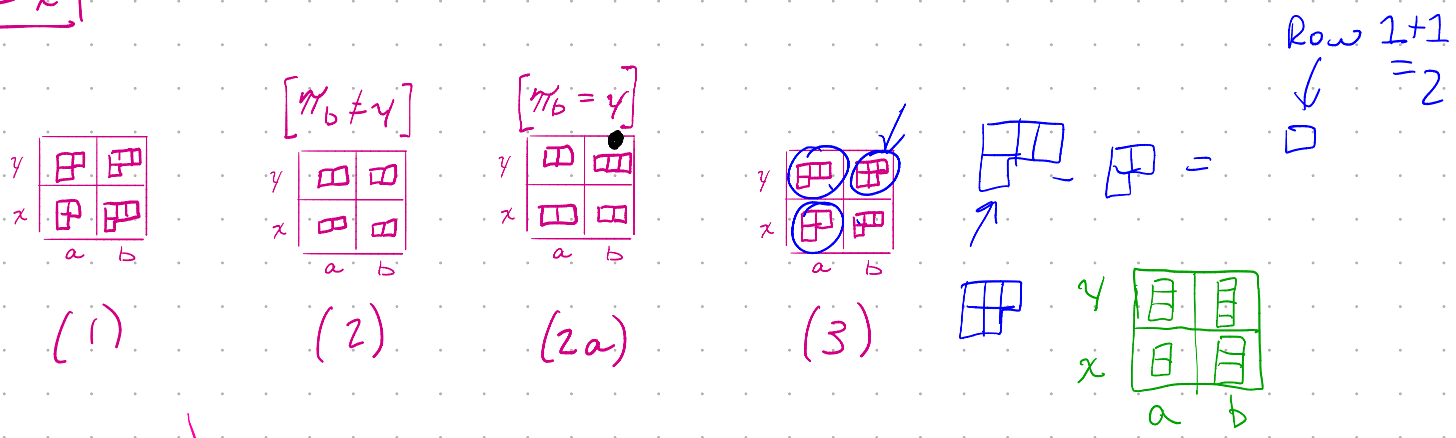
(1) $\lambda_{ay} \neq \lambda_{bx} \Rightarrow \lambda_{by} = \lambda_{ay} \cup \lambda_{bx}$

(2) $\lambda_{ax} = \lambda_{ay} = \lambda_{bx}$ and $\pi_b \neq y \Rightarrow \lambda_{by} = \lambda_{ax}$

(2b) ——— " ——— $\pi_b = y \Rightarrow \lambda_{by} = \lambda_{ax} + \square$ to row 1

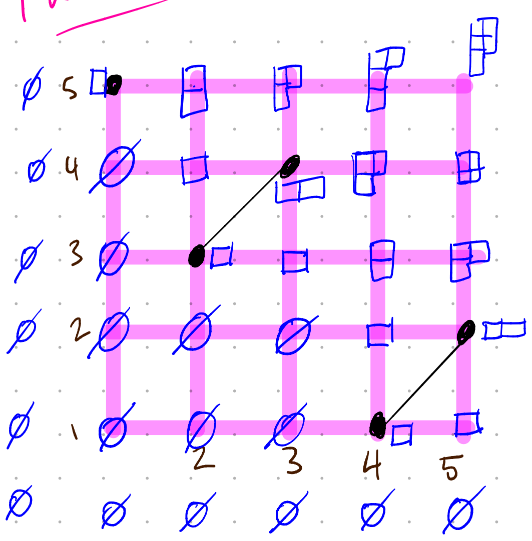
(3) $\lambda_{bx} = \lambda_{ay} \neq \lambda_{ax} \Rightarrow \lambda_{by} = \lambda_{ay} + \square$ row $\ell$ where row $\ell$ = row below $\lambda_{ay} - \lambda_{ax}$

## Ex




## Full Ex

Consider $\pi = 53412$



# Proofs

**Prop 1** (a) The number of boxes in $\lambda_{i+1,j}$ increases by 1 from $\lambda_{ij}$ iff an element of $P$ lies weakly below $(i+1, j)$.

(b) Similarly, the number of boxes in $\lambda_{i,j+1}$ increases iff an element of $P$ lies weakly left of $(i, j+1)$.

Also, we cannot decrease in # of boxes going right to left

Pf: clear by construction.

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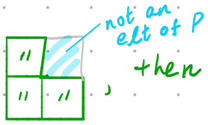
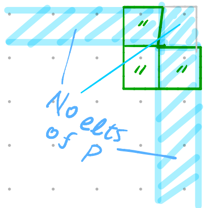
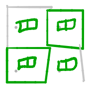
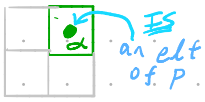
(1) Recursive Growth Thm ✓

(2) If 

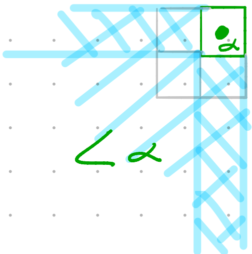
, then 

, by prop 1. ✓

Then $P(a,x) = P(b,y)$, so . QED

(2b) But what if 

? By the original construction of $P$, every elt to the southwest of $\alpha$ is less than $\alpha$.

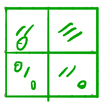
I.e., 

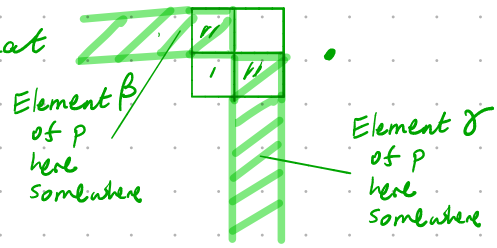
. Thus, any chain in $P(a,x)$ can be ✓

extended by $\alpha$. Therefore, the length of a maximal length chain is increased by 1 going from $P(a,x)$ to $P(b,y)$.

So $\lambda_1(P(b,y)) = \lambda_1(P(a,x)) + 1$, and a box is added to the first row of $\lambda_{ax}$. QED

(3) This proof is a bit more in-depth.

By assumption the setup is , and by prop. 1,

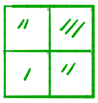
we have that 

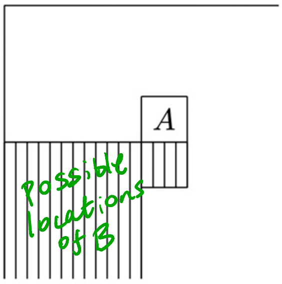
.

# (3) cont'd

Note that $\beta$ and $\gamma$ must be maximal elements in $P(b,y)$.
Also observe that $\lambda(P(b,y)-\{\beta\}) = \underbrace{\lambda_{bx} = \lambda_{ay}}_{\textbf{By assumption}} = \lambda(P(b,y)-\{\gamma\})$.

Now define boxes $A$ and $B$ so that:

if , $B =$ /// − //, $A =$ // − / and apply the GRT (since we are removing 2 maximal elements.)




(a)

Define a poset $P'$ that uses a "northwest" rather than "northeast" ordering rule. Note that, now, chains are antichains and vice-versa.

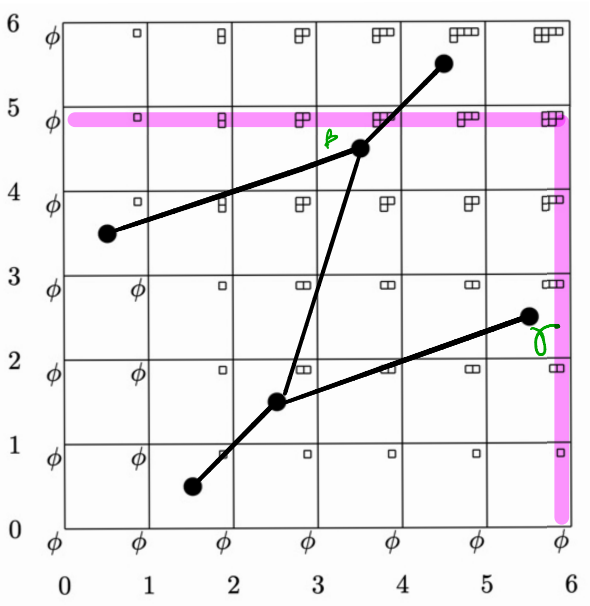



FIGURE 12. The growth diagram for $\sigma = 412563$

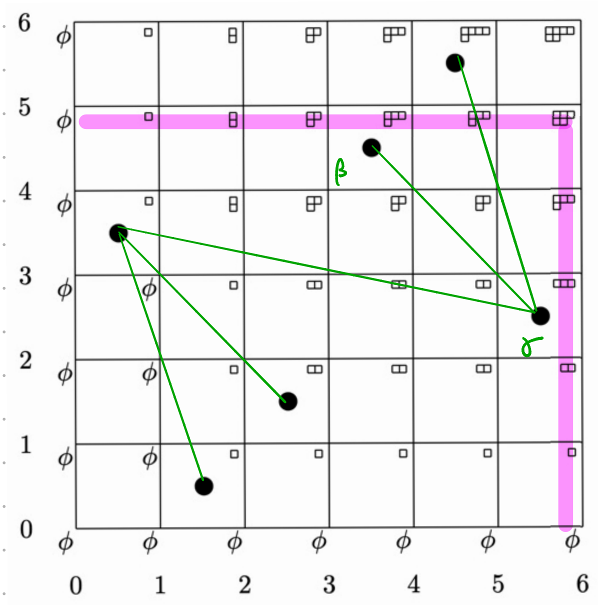
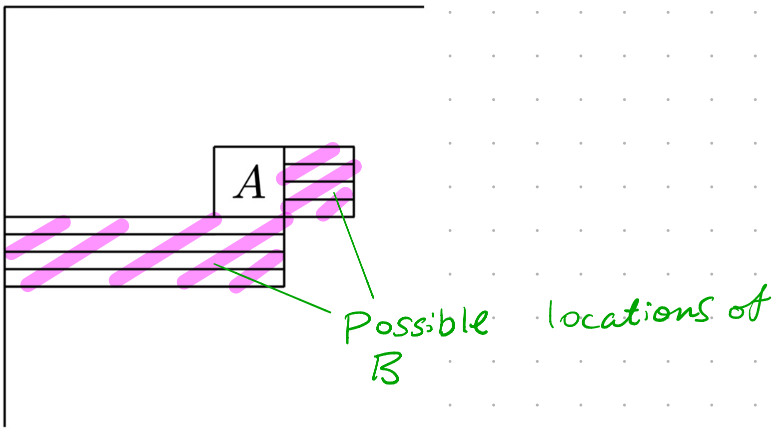


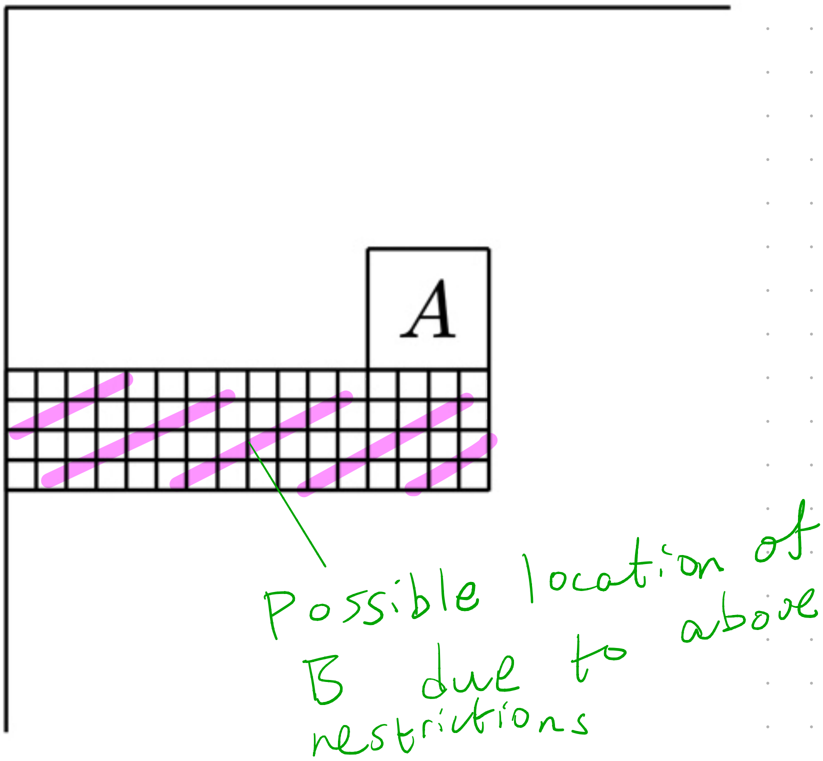

FIGURE 12. The growth diagram for $\sigma = 412563$

Thus, $\lambda(P') = \text{transpose}(\lambda(P))$.
Observe that in $P^{(i,j)}$, both $\beta$ and $\gamma$ are maximal.
However, in $P'^{(i,j)}$, $\beta$ is maximal but $\gamma$ **is minimal.**
Thus, by the GRT, $B$ must lie in the region below.

↓ ↓ ↓




Thus, combining the restrictions, we have that B lies 1 row below A:

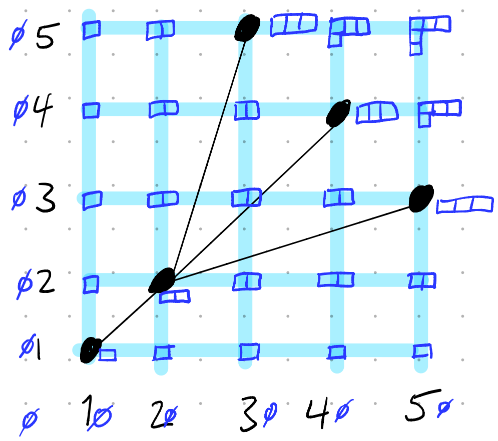



This completes the proof.

# Proving the R.S.-correspondence

**Remark** | We can reconstruct the P- and Q- tableaux from a growth diagram!

Another example: $\pi = 12543$




Running Insertion Tableau

P: $\emptyset \to 1 \to 12 \to 123 \to \begin{smallmatrix}1&2&3\\4\end{smallmatrix} \to \begin{smallmatrix}1&2&3\\4\\5\end{smallmatrix}$

Q: $\emptyset \to \cdots \to \begin{smallmatrix}1&2&3\\4\\5\end{smallmatrix}$

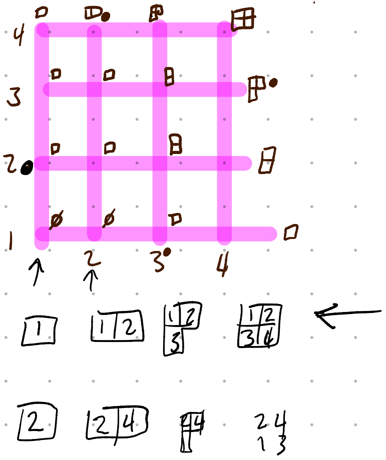
"Normal" algorithm

$\emptyset \leftarrow 1 \quad \boxed{1} \leftarrow 2 \quad \boxed{1\,|\,2} \leftarrow 5 \quad \boxed{1\,|\,2\,|\,5} \leftarrow 4 \quad \begin{smallmatrix}1&2&4\\5\end{smallmatrix} \leftarrow 3 \quad \begin{smallmatrix}1&2&3\\4\\5\end{smallmatrix}$

★ P-tableaux are equal! ★

**Important ★★★ Corollary** | Since each cell in the growth diagram is uniquely determined by its corresponding "triplet", which furthermore determines if there is a point in the cell, we may recover the original permutation, starting at the northeastern boundary of the growth diagram. (I.e., using P & Q as inputs). Thus, we have a bijection between perms. in $S_n$ & pairs of standard tableaux of the same shape.

(yay!)



$$P = \begin{array}{|c|c|}\hline 1 & 3 \\ \hline 2 & 4 \\ \hline \end{array} \qquad \begin{array}{|c|c|}\hline 1 & 2 \\ \hline 3 & 4 \\ \hline \end{array} = Q$$