to the positive root $\alpha_i + \alpha_{i+1} + \cdots + \alpha_j$. Note that $i \leq j$. It is easy to verify that if α corresponds to the node at the intersection of the ith column and the jth row and β corresponds to the node at the intersection of the i'th column and the j'th row, then $(\alpha, \beta) = 1 \Leftrightarrow$ (either $i = i'$ or $j = j'$) and $(i, j) \neq (i', j')$. In other words, two distinct positive roots α and β have a scalar product equal to 1 if and only if the corresponding distinct nodes are in the same row or same column.

We will represent the function $\pi_{\mathbf{i}}$ (respectively $\lambda_{\mathbf{i}}$) by replacing the node corresponding to the root $\alpha \in R(w)$ in the staircase diagram by the value $\pi_{\mathbf{i}}(\alpha)$ (respectively $\lambda_{\mathbf{i}}(\alpha)$) and by leaving the nodes not corresponding to a root of $R(w)$.

EXAMPLE 1.19. Consider the reduced expression $\mathbf{i} = (1, 4, 3, 4, 2, 3, 4, 1, 2, 3)$ of the permutation

$$w_0 = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 \\ 5 & 4 & 3 & 2 & 1 \end{pmatrix} \in S_5.$$

We can describe both $\pi_{\mathbf{i}}$ and $\lambda_{\mathbf{i}}$ using the above convention

$$\pi_{\mathbf{i}} = \begin{array}{cccc} 1 & & & \\ 7 & 10 & & \\ 6 & 9 & 4 & \\ 5 & 8 & 3 & 2 \end{array} \qquad \text{and} \qquad \lambda_{\mathbf{i}} = \begin{array}{cccc} 1 & & & \\ 5 & 6 & & \\ 4 & 5 & 3 & \\ 3 & 4 & 2 & 1 \end{array}$$

Then, by applying Theorem 1.14 repeatedly, we can compute the cardinality of the commutation class $[\mathbf{i}]$. We have underlined the value of the level functions on the top roots to identify these top roots

$$N\begin{pmatrix} 1 & & & \\ 5 & \underline{6} & & \\ 4 & 5 & 3 & \\ 3 & 4 & 2 & 1 \end{pmatrix} = N\begin{pmatrix} 1 & & & \\ \underline{5} & \cdot & & \\ 4 & \underline{5} & 3 & \\ 3 & 4 & 2 & 1 \end{pmatrix} = N\begin{pmatrix} 1 & & & \\ \cdot & \cdot & & \\ 4 & \underline{5} & 3 & \\ 3 & 4 & 2 & 1 \end{pmatrix} + N\begin{pmatrix} 1 & & & \\ \underline{5} & \cdot & & \\ 4 & \cdot & 3 & \\ 3 & \underline{4} & 2 & 1 \end{pmatrix}$$

$$= 2N\begin{pmatrix} 1 & & & \\ \cdot & \cdot & & \\ \underline{4} & \cdot & 3 & \\ 3 & \underline{4} & 2 & 1 \end{pmatrix} + N\begin{pmatrix} 1 & & & \\ \underline{5} & \cdot & & \\ 4 & \cdot & 3 & \\ 3 & \cdot & 2 & 1 \end{pmatrix}$$

$$= 2N\begin{pmatrix} 1 & & & \\ \cdot & \cdot & & \\ \cdot & \cdot & \underline{3} & \\ 3 & \underline{4} & 2 & 1 \end{pmatrix} + 3N\begin{pmatrix} 1 & & & \\ \cdot & \cdot & & \\ \underline{4} & \cdot & 3 & \\ \underline{3} & \cdot & 2 & 1 \end{pmatrix}$$

$$= 2N\begin{pmatrix} 1 & & & \\ \cdot & \cdot & & \\ \cdot & \cdot & \cdot & \\ 3 & \underline{4} & 2 & 1 \end{pmatrix} + 5N\begin{pmatrix} 1 & & & \\ \cdot & \cdot & & \\ \cdot & \cdot & \underline{3} & \\ \underline{3} & \cdot & 2 & 1 \end{pmatrix}$$

$$= 7N\begin{pmatrix} 1 & & & \\ \cdot & \cdot & & \\ \cdot & \cdot & \cdot & \\ \underline{3} & \cdot & 2 & 1 \end{pmatrix} + 5N\begin{pmatrix} \underline{1} & & & \\ \cdot & \cdot & & \\ \cdot & \cdot & \underline{3} & \\ \cdot & \cdot & 2 & 1 \end{pmatrix}$$

$$= 12N\begin{pmatrix} \underline{1} & & & \\ \cdot & \cdot & & \\ \cdot & \cdot & \cdot & \\ \cdot & \cdot & \underline{2} & 1 \end{pmatrix} + 5N\begin{pmatrix} \cdot & & & \\ \cdot & \cdot & & \\ \cdot & \cdot & \underline{3} & \\ \cdot & \cdot & 2 & 1 \end{pmatrix}$$

$$= 17N \begin{pmatrix} \cdot & & & \\ \cdot & \cdot & & \\ \cdot & \cdot & \cdot & \\ \cdot & \cdot & \underline{2} & 1 \end{pmatrix} + 12N \begin{pmatrix} \underline{1} & & & \\ \cdot & \cdot & & \\ \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \underline{1} \end{pmatrix}$$

$$= 29N \begin{pmatrix} \cdot & & & \\ \cdot & \cdot & & \\ \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \underline{1} \end{pmatrix} + 12N \begin{pmatrix} \underline{1} & & & \\ \cdot & \cdot & & \\ \cdot & \cdot & \cdot & \\ \cdot & \cdot & \cdot & \cdot \end{pmatrix} = 29 + 12 = 41.$$

Thus the commutation class $[\mathbf{i}]$ has 41 elements.

2. Reduced Expressions Adapted to a Quiver

2.1. In this section, we will assume that the root system R is irreducible and simply laced. In particular, the Cartan matrix C is symmetric and positive definite. We will use the same convention as in 1.15. Δ will denote the Dynkin graph of R. Here Δ is connected and the set of vertices of Δ is $\{1, 2, \dots, n\}$ where i is identified with the simple root $\alpha_i \in B$.

It is well known that there exists a unique element w_0 of the Weyl group W that is of maximal length and, in this case $\ell(w_0) = \#(R^+)$. We will also denote this length by ν.

Let σ be the unique permutation of the vertices of Δ such that $w_0(\alpha_i) = -\alpha_{\sigma(i)}$. In other words, if Δ is of type D_n with n even or of type A_1, E_7 or E_8, then σ is the identity; while if Δ is of type A_n with $n > 1$, D_n with n odd or E_6, then σ is the unique non-trivial automorphism of the graph Δ. Denote by h, the Coxeter number of Δ. In other words, h is $n+1$, $2(n-1)$, 12, 18 or 30, if Δ is respectively of type A_n, D_n, E_6, E_7 or E_8.

2.2. Given a graph G whose edges are oriented, we say that a vertex i is a sink (respectively a source) if and only if each edge $\{i, j\}$ having i as one of its vertices is oriented as follows: $i \leftarrow j$, the arrow pointing towards i (respectively $i \rightarrow j$, the arrow pointing away from i).

2.3. We will recall the notation of Section 4 of [2] for the theory of representations of quiver. Let Ω be a quiver with underlying graph Δ. In other words, we have oriented the edges of Δ. Let F be a fixed field. Recall that if i is a sink (respectively a source) of Ω, then:

(a) $s_i(\Omega)$ denote the quiver obtained from Ω by reversing the orientation of each arrow that ends (respectively starts) at i;

(b) Φ_i^+ (respectively Φ_i^-) denote the corresponding reflection functor from the category of modules of Ω to the category of modules of $s_i(\Omega)$.

2.4. A reduced expression $\mathbf{i} = (i_1, i_2, \dots, i_\nu)$ of w_0 is said to be adapted to the quiver Ω if and only if i_k is a sink of $s_{i_{k-1}} s_{i_{k-2}} \cdots s_{i_1}(\Omega) = \Omega_k$ for all $k = 1, 2, \dots, \nu$. For example, the reduced expression $\mathbf{i}$ of Example 1.19 is adapted to the quiver $1 \leftarrow 2 \rightarrow 3 \rightarrow 4$.

The following facts are known:

(a) A reduced expression $\mathbf{i}$ of w_0 is adapted to at most one quiver Ω of Δ.

(b) For each quiver Ω with graph Δ, there is a reduced expression $\mathbf{i}$ of w_0 adapted to Ω.

(c) Let $\mathbf{i}, \mathbf{j}$ be two reduced expressions of w_0 such that $\mathbf{j} \sim \mathbf{i}$. If $\mathbf{i}$ is adapted to the quiver Ω with graph Δ, then so is $\mathbf{j}$.

For (a), see 4.13 in [2]. For (b), see Proposition 4.12 (b) in [2]. As for (c), it is easy to verify by simply considering the case where $\mathbf{j}$ is obtained from $\mathbf{i}$ by doing a short braid relation.

Note that it is a special property of a reduced expression to be adapted to a quiver. There are reduced expressions that are not adapted to any quiver. For example in the case