Pg 4–6 of Stanley's "Promotion and Evacuation"

**Thm 2.1**

(a) $\epsilon^2 = 1$ (the identity operator)

(b) $\partial^p = \epsilon\epsilon^*$

(c) $\partial\epsilon = \epsilon\partial^{-1}$

**Proof of Thm 2.1**

Regard $f \in \mathcal{L}(P)$ as the word $f^{-1}(1), \dots, f^{-1}(p)$.

Define $\tau_i : \mathcal{L}(P) \longrightarrow \mathcal{L}(P)$ for $1 \le i \le p-1$ by

$$\tau_i(u_1 u_2 \dots u_p) = \begin{cases} u_1 u_2 \cdots\cdots u_p & \text{if } u_i \,\&\, u_{i+1} \text{ are comparable in } P \\ u_1 u_2 \cdots u_{i+1} u_i \cdots u_p & \text{otherwise} \end{cases}$$

The $\tau_i$ are bijections, and they satisfy the relations

$$\tau_i^2 = 1, \quad 1 \le i \le p-1$$

$$\tau_i \tau_j = \tau_j \tau_i, \quad \text{if } |i-j| > 1.$$

Let $\gamma := \tau_1 \tau_2 \cdots \tau_{p-1} \cdot \tau_1 \tau_2 \cdots \tau_{p-2} \cdot \cdots \tau_1 \tau_2 \cdot \tau_1$

$\gamma^* := \tau_{p-1} \tau_{p-2} \cdots \tau_1 \cdot \tau_{p-1} \tau_{p-2} \cdots \tau_2 \cdot \cdots \tau_{p-1} \tau_{p-2} \cdot \tau_{p-1}$

- Note that $f\epsilon$ is defined to be (Read left to right)

| | | |
|---|---|---|
| step 1 | Get $f\partial$ by applying $\partial$ to $f$ | $\tau_1 \tau_2 \tau_3 \dots \tau_{p-1}$ |
| step 2 | Freeze label $p$, then apply $\partial$ to $f\partial - \{p\}$ | $\tau_1 \tau_2 \tau_3 \cdots \tau_{p-2}$ |
| step 3 | Freeze label $p-1$, then apply $\partial$ to $(f\partial - \{p\})\partial$ | $\tau_1 \tau_2 \tau_3 \cdots \tau_{p-3}$ |
| $\vdots$ | | |
| Step p-1 | Freeze label 3, then apply $\partial$ to the remaining 2-elt poset | $\tau_1$ |

So, we have $\epsilon = \gamma$.

- Similarly, $\epsilon^* = \gamma^*$

Let $\delta := \tau_1 \tau_2 \cdots \tau_{p-1}$.

Page 5–6 show that $\partial = \delta$.

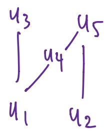
**Lemma 2.2**

(a) $\gamma^2 = (\gamma^*)^2 = 1$

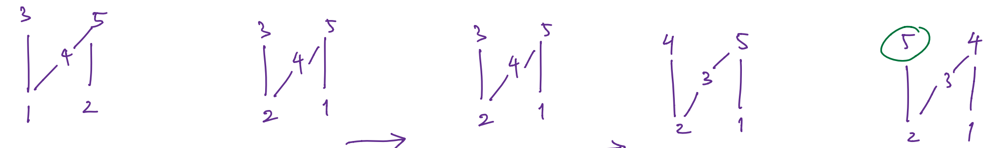
(b) $\delta^p = \gamma\gamma^*$

(c) $\delta\gamma = \gamma\delta^{-1}$

See example

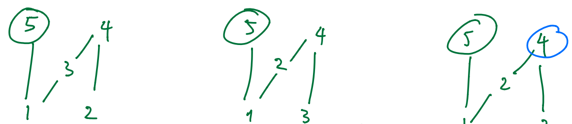
$P=$ 

From Fig. 2



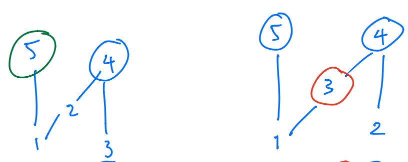
$f = u_1u_2u_3u_4u_5 \xrightarrow{T_1} u_2u_1u_3u_4u_5 \xrightarrow{T_2} u_2u_1u_3u_4u_5 \xrightarrow{T_3} u_2u_1u_4u_3u_5 \xrightarrow{T_4} u_2u_1u_4u_5u_3$

(do nothing because $u_1$ & $u_3$ are comparable)



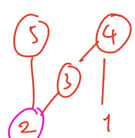
$\xrightarrow{T_1} u_1u_2u_4u_5u_3 \xrightarrow{T_2} u_1u_4u_2u_5u_3 \xrightarrow{T_3} u_1u_4u_2u_5u_3$

do nothing because $u_2, u_5$ are comparable



$\xrightarrow{T_1} u_1u_4u_2u_5u_3 \xrightarrow{T_2} u_1u_2u_4u_5u_3$

do nothing because $u_1$ and $u_4$ are comparable



$\xrightarrow{T_1} u_2u_1u_4u_5u_3$

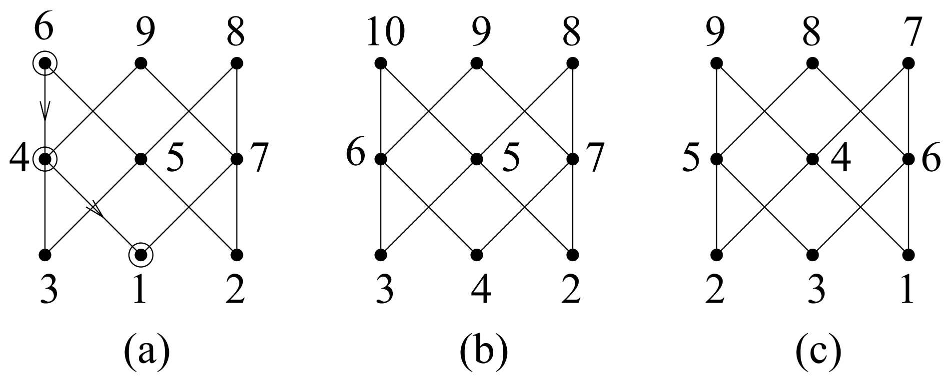


Figure 1: The promotion operator $\partial$ applied to a linear extension

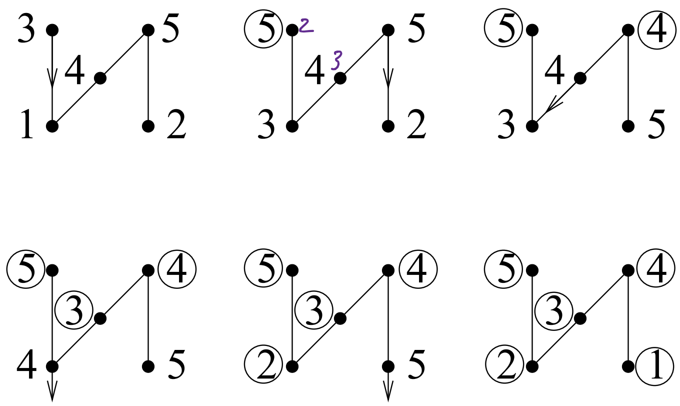


Figure 2: The evacuation of a linear extension $f$

Then "freeze" the label $p$ into place and apply $\partial$ to what remains. In other words, let $P_1$ consist of those elements of $P$ labelled $1, 2, \ldots, p-1$ by $f\partial$, and apply $\partial$ to the restriction of $\partial f$ to $P_1$. Then freeze the label $p-1$ and apply $\partial$ to the $p-2$ elements that remain. Continue in this way until every element has been frozen. Let $f\epsilon$ be the linear extension, called the *evacuation* of $f$, defined by the frozen labels.

NOTE. A standard Young tableau of shape $\lambda$ can be identified in an obvious way with a linear extension of a certain poset $P_\lambda$. Evacuation of standard Young tableaux has a nice geometric interpretation connected with the nilpotent flag variety. See van Leeuwen [18, §3] and Tesler [36, Thm. 5.14].

Figure 2 illustrates the evacuation of a linear extension $f$. The promotion paths are shown by arrows, and the frozen elements are circled. For ease of understanding we don't subtract 1 from the unfrozen labels since they all eventually disappear. The labels are always frozen in descending order $p, p-1, \ldots, 1$. Figure 3 shows the evacuation of $f\epsilon$, where $f$ is the linear extension of Figure 2. Note that (seemingly) miraculously we have $f\epsilon^2 = f$. This example illustrates a fundamental property of evacuation given by Theorem 2.1(a) below.

We can define *dual evacuation* analogously to dual promotion. In symbols, if $f \in \mathcal{L}(P)$