2.10 c-sorting word

Def of c-sorting word

Given a Coxeter element $c \in S_{n+1}$, fix a reduced word $c = a_1 a_2 \cdots a_n$. where $a_k \in \{s_1, \dots, s_n\}$

Let $c^\infty := c | c | c | \cdots$
$= a_1 a_2 \cdots a_n | a_1 a_2 \cdots a_n | a_1 a_2 \cdots a_n | \cdots$

Given $\pi \in S_{n+1}$, the $(a_1, a_2, \dots, a_n)$-sorting word for π is the subword of c^∞ which is lexicographically first (as a sequence of positions in c^∞) and is a reduced word for π.

- Every permutation has exactly one c-sorting word.
The c-sorting word depends on the choice of reduced word $a_1 a_2 \dots a_n$.

Example 1 Fix $c = s_1 s_3 s_2$. Note: this Coxeter elt c has two different reduced words. Here I use $s_1 s_3 s_2$.

$c^\infty = s_1 s_3 s_2 | s_1 s_3 s_2 | s_1 s_3 s_2 | s_1 s_3 s_2 | \cdots$

(i) $\pi = 1432 = s_3 s_2 s_3 = s_2 s_3 s_2$ has two reduced words.

Comparing $s_1 s_3 s_2 | s_1 s_3 s_2 | s_1 s_3 s_2 | s_1 s_3 s_2 | \cdots$ and
$s_1 s_3 s_2 | s_1 s_3 s_2 | s_1 s_3 s_2 | s_1 s_3 s_2 | \cdots$,

the reduced word $s_3 s_2 s_3$ is lexicographically first (as a sequence of positions in c^∞)
so the c-sorting word for 1432 is $s_3 s_2 s_3$.

(ii) $w = 4132 = s_3 s_2 s_3 s_1 = s_2 s_3 s_2 s_1 = s_3 s_2 s_1 s_3$ has three reduced words.

Comparing $s_1 s_3 s_2 | s_1 s_3 s_2 | s_1 s_3 s_2 | s_1 s_3 s_2 | \cdots$,
$s_1 s_3 s_2 | s_1 s_3 s_2 | s_1 s_3 s_2 | s_1 s_3 s_2 | \cdots$, and
$s_1 s_3 s_2 | s_1 s_3 s_2 | s_1 s_3 s_2 | s_1 s_3 s_2 | \cdots$,

the reduced word $s_3 s_2 s_1 s_3$ is the c-sorting word for w.

Algorithm for finding the $(a_1, a_2, \dots, a_n)$-sorting word of a permutation π, assuming we already know the length of π, and $\pi \neq \mathrm{Id}$. Ref: "Comb. Lattices & Beyond"

We write down a reduced word $\pi = u_1 u_2 \cdots u_\ell$ as follows, where $u_i \in S$.

- First, try each letter $a_1, a_2, \dots, a_n$ (in this order) until we find one a_i s.t $\ell(a_i \pi) < \ell(\pi)$.
Take a_i to be the first letter for π, set $u_1 := a_i$, and write $\pi' = u_1 \pi$.
- If $\pi' = \mathrm{Id}$, then $\ell = 1$ and u_1 is the desired $(a_1, a_2, \dots, a_n)$-sorting word.

Otherwise, try each of the n letters in the order $a_{i+1}, a_{i+2}, \dots, a_n, a_1, \dots$ until we find one $a_{i'}$ s.t $\ell(a_{i'} \pi') < \ell(\pi')$. Take $a_{i'}$ to be the second letter for π, set $u_2 := a_{i'}$, and define $\pi'' = u_2 \pi'$.

- If $\pi'' = \mathrm{Id}$, then $\ell = 2$ and $u_1 u_2$ is the desired $(a_1, a_2, \dots, a_n)$-sorting word.
Otherwise, continuing in this manner, we eventually find a reduced word for π.
This is the $(a_1, a_2, \dots, a_n)$-sorting word for π.

Example 2

Example of computing a c-sorting word (Ref: Thesis by Suleiman)

For $c = \overset{a_1}{s_1}\overset{a_2}{s_2}\overset{a_3}{s_3}\overset{a_4}{s_4}$

(i) Let $\pi = s_1 s_4 s_3 s_4 = (12)(35) = 21543$, and assume we know (for example by computing inv(π) with Sage) that length(π) = 4

$$s_1\, s_2\, s_3\, s_4 \mid s_1\, s_2\, s_3\, s_4 \mid s_1\, s_2\, s_3\, s_4$$

- $u_1 = s_1$ because $s_1 s_1 = Id$. Set $\pi' = u_1\pi = s_1 s_1 s_4 s_3 s_4 = s_4 s_3 s_4$, of length 3
- Next, try each of $\underbrace{s_2, s_3, s_4, s_1}$, the next n letters in c^∞ after u_1,

$$s_1\, \overbrace{s_2\, s_3\, s_4 \mid s_1}\, s_2\, s_3\, s_4 \mid s_1\, s_2\, s_3\, s_4$$

until we find s_i s.t $\ell(s_i\pi') < \ell(\pi')$

* Try s_2: $s_2\pi' = s_2 s_4 s_3 s_4$ has length 4 (To see this, we can check the number of inversions of π')
Doesn't work — keep trying

* Try s_3: $s_3\pi' = s_3 s_4 s_3 s_4$
$= s_4 s_3 s_4 s_4$ by the long braid move
$= s_4 s_3$ since $s_4 s_4 = Id$
has length 2, smaller than $3 = \ell(\pi')$

So we set $u_2 = s_3$, and set $\pi'' = u_2\pi'$
$= s_3 s_4 s_3 s_4$
$= s_4 s_3$

$$s_1\, s_2\, s_3\, s_4 \mid s_1\, s_2\, s_3\, s_4 \mid s_1\, s_2\, s_3\, s_4$$

- Next, try each of $\underbrace{s_4, s_1, s_2, s_3}$ the next n letters in c^∞ after u_2

$$s_1\, s_2\, s_3\, \overbrace{s_4 \mid s_1\, s_2\, s_3}\, s_4 \mid s_1\, s_2\, s_3\, s_4$$

until we find s_i s.t $\ell(s_i\pi'') < \ell(\pi'')$.

Then $u_3 = s_4$ because $s_4\pi'' = s_4 s_4 s_3 = s_3$ has length 1. Set $\pi''' = s_3$

$$s_1\, s_2\, s_3\, s_4 \mid s_1\, s_2\, s_3\, s_4 \mid s_1\, s_2\, s_3\, s_4$$

- Next, $u_4 = s_3$ since $\pi''' = s_3$. ∴ The (s_1, s_2, s_3, s_4)-sorting word of π is $s_1 s_3 s_4 s_3$.

(ii) $\boxed{\text{PRACTICE}}$ Verify this example

Let $w = s_2 s_1 = (23)(12) = (132) = 31245$.
Following the algorithm, we get $s_2 s_1$ as the (s_1, s_2, s_3, s_4)-sorting word of w.

$$s_1\, s_2\, s_3\, s_4 \mid s_1\, s_2\, s_3\, s_4 \mid s_1\, s_2\, s_3\, s_4.$$

(iii) $\boxed{\text{PRACTICE}}$ Verify this example

Let $z = s_1 s_3 s_2 = s_3 s_1 s_2 = (1243) = 24135$.
Following the algorithm, we get $s_1 s_3 s_2$ as the (s_1, s_2, s_3, s_4)-sorting word of z.

$$s_1\, s_2\, s_3\, s_4 \mid s_1\, s_2\, s_3\, s_4 \mid s_1\, s_2\, s_3\, s_4.$$

2.11 c-sortable words

<u>Def of c-sortable word</u> Let c be a Coxeter element, and $a_1 a_2 a_3 \cdots a_n$ be a reduced word of c.

For a set $K = \{ i_1 < i_2 < \cdots < i_r \} \subset [n]$, let

c_K denote $a_{i_1} a_{i_2} \cdots a_{i_r}$.

Ex $c = (a_1, a_2, a_3, a_4)$
$K = \{1, 2, 4\}$
$c_K = a_1 a_2 a_4$

The c-sorting word for a permutation $\pi \in S_{n+1}$ can be uniquely written as

$$\pi = c_{K_1} c_{K_2} \cdots c_{K_p}, \quad \text{where } p \text{ is minimal.}$$

If $K_1 \supset K_2 \supset \cdots \supset K_p$, we say that

π is <u>c-sortable</u> and $c_{K_1} c_{K_2} \cdots c_{K_p}$ is a <u>c-sortable word</u>.

- Not every permutation is c-sortable.

<u>Fact</u> Although the def of c-sortable requires a choice of a reduced word of c,

the set of c-sortable elements does not depend on the choice of reduced word for c.

From above Example 1,

for $c = s_1 s_3 s_2 = s_3 s_1 s_2$

c has two reduced words – here I choose $s_1 s_3 s_2$.
But whether or not a permutation is c-sortable does not depend on my choice of reduced words.

(i) $\pi = 1432 = s_3 s_2 s_3 = s_1 s_3 s_2 \,|\, s_1 s_3 s_2 \,|\, s_1 s_3 s_2 \,|\, s_1 s_3 s_2 \,|\, \cdots$

$\{s_3, s_2\} \supset \{s_3\} \supset \varnothing \cdots$

is the c-sorting word for π, so π is c-sortable,

(ii) $w = 4132 = s_3 s_2 s_1 s_3 = s_1 s_3 s_2 \,|\, s_1 s_3 s_2 \,|\, s_1 s_3 s_2 \,|\, s_1 s_3 s_2 \,|\, \cdots$

$\{s_3, s_2\} \not\supset \{s_1\}$

is the c-sorting word for w, so w is not c-sortable.

From above example 2, for $c = s_1 s_2 s_3 s_4$

(i) $\pi = s_1 s_3 s_4 s_3$ $\quad s_1 s_2 s_3 s_4 \,|\, s_1 s_2 s_3 s_4 \,|\, s_1 s_2 s_3 s_4$ is c-sortable.

$\{s_1, s_3, s_4\} \supseteq \{s_3\} \supseteq \varnothing \cdots$

(ii) $w = s_2 s_1$ $\quad s_1 s_2 s_3 s_4 \,|\, s_1 s_2 s_3 s_4 \,|\, s_1 s_2 s_3 s_4$

$\{s_2\} \not\supseteq \{s_1\}$

(iii) $z = s_1 s_3 s_2$ $\quad s_1 s_2 s_3 s_4 \,|\, s_1 s_2 s_3 s_4 \,|\, s_1 s_2 s_3 s_4$

$\{s_1, s_3\} \not\supseteq \{s_2\}$

(ii), (iii): not c-sortable

$\boxed{\text{Thm 2.11}}$ Let c be any Coxeter elt.

- The identity permutation is c-sortable (since it is the empty word).
- The longest elt w_0 is c-sortable.

<u>Example 3</u> <u>for Thm 2.11</u>

Apply the algorithm 2.10 for A_3, $w_0 = 4321$. We know w_0 has length $\binom{4}{2} = 6$.

For $c = s_1 s_2 s_3$ (Tamari Coxeter element)

<u>Note:</u> Multiplying $s_i = (i, i{+}1)$ on the left corresponds to swapping values $i, i{+}1$

First, apply s_1: $s_1 w_0 = s_1 \circ 4321 \overset{\text{swaps } 2,1}{=} 4312$.

Set $u_1 = s_1$, and $w_0' := s_1 w_0 = 4312$

$s_1\ s_2\ s_3 \mid s_1\ s_2\ s_3 \mid s_1\ s_2\ s_3$

Try the next letter in c^∞, which is s_2: $s_2 w_0' = s_2 \circ 4312 \overset{\text{swaps } 3,2}{=} 4213$

This puts 2,3 in order, which decreases the inversion number,

so $\ell(s_2 w_0') < \ell(w_0')$.

Set $u_2 = s_2$, and $w_0'' := s_2 w_1' = 4213$

$s_1\ s_2\ s_3 \mid s_1\ s_2\ s_3 \mid s_1\ s_2\ s_3$

Try the next letter in c^∞, which is s_3: $s_3 w_0'' = s_3 \circ 4213 \overset{\text{swaps } 4,3}{=} 3214$

Again, $\ell(s_3 \circ w_0'') < \ell(w_0'')$.

Set $u_3 = s_3$, and $w_0''' := s_3 w_0'' = 3214$

$s_1\ s_2\ s_3 \mid s_1\ s_2\ s_3 \mid s_1\ s_2\ s_3$

The next letter in c^∞, s_1, also works because entries 2,1 in $w_0''' = 3\underbrace{21}4$ are out of order.

Set $\underline{u_4 = s_1}$, and $w_0^{(4)} := s_1 w_0''' = 3124$.

The next letter in c^∞, s_2, also works because entries 3,2 in $w_0^{(4)} = \underbrace{31}24$ are out of order.

Set $\underline{u_5 = s_2}$, and $w_0^{(5)} := s_2 w_0^{(4)} = 2134$.

$s_1\ s_2\ s_3 \mid s_1\ s_2\ s_3 \mid s_1\ s_2\ s_3$

The next letter in c^∞, s_3, does <u>not</u> work because $s_3 w_0^{(5)} = s_3 \circ 21\underbrace{34}_{\substack{3,4\\ \text{in}\\ \text{order}}} = 21\underbrace{43}_{\substack{3,4\\ \text{out of}\\ \text{order}}}$

$\rightarrow$ The next letter in c^∞ after s_3 is s_1:

s_1 works because $s_1 \circ w_0^{(5)} = s_1 \circ 2134 \overset{\text{swaps } 2,1}{=} 1234$.

Set $u_6 = s_1$

<u>Note:</u> $w_0 = s_1 s_2 s_3 s_2 s_1 s_2$ is not the (s_1, s_2, s_3)-sorting word

$s_1\ s_2\ s_3 \mid s_1\ s_2\ s_3 \mid s_1\ s_2\ s_3$

$\therefore$ The (s_1, s_2, s_3)-sorting word for $w_0 = 4321$ is $s_1 s_2 s_3 s_1 s_2 s_1$.

Since $\{s_1, s_2, s_3\} \supset \{s_1, s_2\} \supset \{s_1\}$, this shows $w_0 = 4321$ is c-sortable.

Example for Thm 2.11

Apply the algorithm 2.10 for A_4, $w_0 = 54321$. We know w_0 has length $\binom{5}{2} = 10$.
For $c = s_3 s_1 s_2 s_4$ (a "bipartite" Coxeter element)

Note: Multiplying $s_i = (i, i+1)$ on the left corresponds to swapping values $i, i+1$

First, try s_3: $s_3 w_0 = s_3 \circ 54321 = 53421$.
Set $u_1 = s_3$, and $w_0' := s_3 w_0 = 53421$

$s_3\ s_1\ s_2\ s_4 \mid s_3\ s_1\ s_2\ s_4 \mid s_3\ s_1\ s_2\ s_4$

Try the next letter in c^∞, which is s_1 : $s_1 w_0' = s_1 \circ 53421 \overset{\text{swap } 1,2}{=} 53412$
This puts 1,2 in order, which decreases the inversion number,
so $\ell(s_1 w_0') < \ell(w_0')$.
Set $u_2 = s_1$, and $w_0'' := s_1 w_0' = 53412$.

$s_3\ s_1\ s_2\ s_4 \mid s_3\ s_1\ s_2\ s_4 \mid s_3\ s_1\ s_2\ s_4$

The next letter in c^∞, s_2, works: $s_2 w_0'' = s_2 \circ 53412 \overset{\text{swaps } 2,3}{=} 52413$
Set $u_3 = s_2$, and $w_0''' := s_2 w_0'' = 52413$.

Next, set
$u_4 = s_4$, and $w_0^{(4)} := s_4 w_0''' = s_4 \circ 52413 = 42513$
$u_5 = s_3$, and $w_0^{(5)} := s_3 w_0^{(4)} = s_3 \circ 42513 = 32514$
$u_6 = s_1$, and $w_0^{(6)} := s_1 w_0^{(5)} = s_1 \circ 32514 = 31524$
$u_7 = s_2$, and $w_0^{(7)} := s_2 w_0^{(6)} = s_2 \circ 31524 = 21534$
$u_8 = s_4$, and $w_0^{(8)} := s_4 w_0^{(7)} = s_4 \circ 21534 = 21435$
$u_9 = s_3$, and $w_0^{(9)} := s_3 w_0^{(8)} = s_3 \circ 21435 = 21345$
$u_{10} = s_1$, and $w_0^{(10)} := s_1 w_0^{(9)} = s_1 \circ 21345 = 12345$

PRACTICE
Verify computation for $u_4, \dots, u_{10}$

$s_3\ s_1\ s_2\ s_4 \mid s_3\ s_1\ s_2\ s_4 \mid s_3\ s_1\ s_2\ s_4$, So the (s_3, s_1, s_2, s_4)-sorting word for 54321 is $s_3\, s_1\, s_2\, s_4\, s_3\, s_1\, s_2\, s_4\, s_3\, s_1$, and so 54321 is $s_3 s_1 s_2 s_4$-sortable.

PRACTICE
Use Algorithm 2.10 to verify these two examples

More examples for type A_4 Coxeter group

- If $c = s_1 s_3 s_5 s_2 s_4$, $w_0 = s_1\, s_3\, s_5\, s_2\, s_4 \mid s_1\, s_3\, s_5\, s_2\, s_4 \mid s_1\, s_3\, s_5\, s_2\, s_4 \mid$ is the c-sorting word of w_0, which shows w_0 is c-sortable.
- If $c = s_1 s_2 s_3 s_4 s_5$, $w_0 = s_1\, s_2\, s_3\, s_4\, s_5 \mid s_1\, s_2\, s_3\, s_4\, s_5 \mid s_1\ 2\ s_3\, s_4\, s_5 \mid s_1\, s_2\, s_3\, s_4\, s_5 \mid s_1\, s_2\, s_3\, s_4\, s_5$ is the c-sorting word of w_0, which shows w_0 is c-sortable.

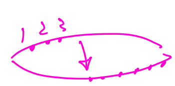
REU Exercise 11

Exercises 8 & 9 give us the c-sorting word (of w_0) for (Tamari) $c = s_1 s_2 \dots s_n$ and (bipartite) $c =$ (odd indices)(even indices)

Now, consider the Coxeter element $c := \underbrace{s_{\lceil \frac{n}{2} \rceil} \dots s_3\, s_2\, s_1}_{\text{the first half of indices}}\ \underbrace{s_{\lceil \frac{n}{2} \rceil + 1}, \dots, s_{n-2}, s_{n-1}, s_n}_{\text{the second half of indices}}$

for example
$c = s_2 s_1 s_3$
$c = s_2 s_1 s_3 s_4$
$c = s_3 s_2 s_1 s_4 s_5$
$c = s_3 s_2 s_1 s_4 s_5 s_6$
$c = s_4 s_3 s_2 s_1 s_5 s_6 s_7$

"Halfway"



Give the c-sorting word of w_0 for for arbitrary n.

<u>Known</u> For Tamari, $\lambda_1 - 4 = \lambda_2$ for $n \geq 4$, for $c = s_1 s_2 \dots s_n$

$\lambda_1 - 2 = \lambda_2$ for $n \geq 4$, for $c = s_1 s_3 s_5 \dots s_2 s_4 \dots$

<u>Conjecture</u> For "halfway" c, $\lambda_1 - 3 = \lambda_2$ for $n \geq 4$.

<u>Thm 2.11 (b)</u>

A permutation π is the minimum elt of its η_c-fiber iff π is c-sortable.

~ end Thu June 18, 2020 ~

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