2.8 Coxeter groups

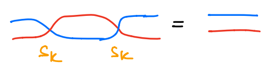
Last updated Thurs, June 25, 2020

Lectured on Tue, June 16, 2020
Thurs, June 18, 2020

We will now think of the symmetric group S_{n+1} as a Coxeter group W of type A_n.

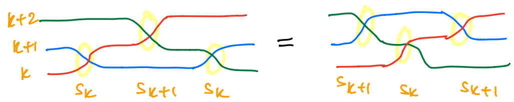
A <u>type A_n Coxeter group W</u> is generated by $S := \{s_1, s_2, \dots, s_n\} \subset W$

where $s_k^2 = \underset{\text{identity elt}}{1}$

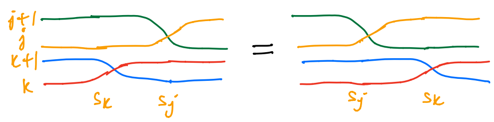


for all $k \in [n]$

$s_k s_{k+1} s_k = s_{k+1} s_k s_{k+1}$ (braid relation, or long braid relation) for all $k \in [n-1]$.



$s_k s_j = s_j s_k$ if $|k-j| \geq 2$ (commutation relation, or short braid relation)



The $s_1, s_2, \dots, s_n$ are called <u>simple reflections</u>.

Every element $\pi \in W$ can be written (non-uniquely) as a <u>word</u> in the alphabet of S, that is, as a product of the simple reflections:

$$\pi = s_{i_1} s_{i_2} \cdots s_{i_\ell}, \qquad s_{i_k} \in S = \{s_1, s_2, \dots, s_n\}.$$

If ℓ is minimal among all words for π, then ℓ is called the <u>length</u> of π, and the word $s_{i_1} s_{i_2} \cdots s_{i_\ell}$ is called a <u>reduced word</u> for π.
(or <u>reduced decomposition</u>, or <u>reduced expression</u>)

<u>Example</u>: $\pi = s_2 s_1 s_2 s_1 s_3 s_2 s_3 \in W$

$= s_1 s_2 s_1 s_1 s_3 s_2 s_3$ by the braid relation

$= s_1 s_2 (s_1 s_1) s_3 s_2 s_3$

$= s_1 s_2 s_3 s_2 s_3$

$= s_1 s_3 s_2 s_3 s_3$ by the braid relation

$= s_1 s_3 s_2$

$= s_3 s_1 s_2$

Both $s_1 s_3 s_2$ and $s_3 s_1 s_2$ are reduced words for π.
$\ell(\pi) = 3$

<u>Facts</u>

$\ell(\pi s) = \ell(\pi) + 1$ or $\ell(\pi) - 1$ for any $s \in S$

$\ell(s\pi) = \ell(\pi) + 1$ or $\ell(\pi) - 1$ for any $s \in S$

If $x \in S$,

$\ell(\pi x) < \ell(\pi)$ iff π has a reduced word which ends with x.

$\ell(x\pi) < \ell(\pi)$ iff π has a reduced word which starts with x.

Example: For $\pi = s_1 s_3 s_2 = s_3 s_1 s_2$,

$$\ell(\pi s_2) < \ell(\pi), \text{ but } \ell(\pi s_k) > \ell(\pi) \text{ for } k \neq 2.$$

Facts

- The map $\varepsilon : s_k \mapsto -1$ for all $s_k \in S$ extends to a group homomorphism

  $$\varepsilon : W \to \{+1, -1\},$$

  so $\varepsilon(\pi) = (-1)^{\ell(\pi)}$

  Half the elements of the Coxeter group W are odd permutations, & half are even permutations.

- $\ell(\pi^{-1}) = \ell(\pi)$

  Example: the inverse of $s_{i_1} s_{i_2} \cdots s_{i_\ell}$ is $s_{i_\ell} \cdots s_{i_2} s_{i_1}$.

- $\ell(w\pi) \leq \ell(w) + \ell(\pi)$

Def The right weak order on W can be defined by $x \leq y$ iff x appears as a prefix of a word of y, that is,

there are $s_{i_1}, s_{i_2}, \ldots, s_{i_j} \in S$ s.t. $y = x\, s_{i_1} s_{i_2} \cdots s_{i_j}$ with $\ell(x) + j = \ell(y)$.

$x \lessdot y$ "is covered by" when $xs = y$ for some $s \in S$ with $\ell(x) + 1 = \ell(y)$.

The common convention is to set $s_1 := (1,2)$, $s_2 := (2,3), \ldots, s_n := (n, n+1)$.

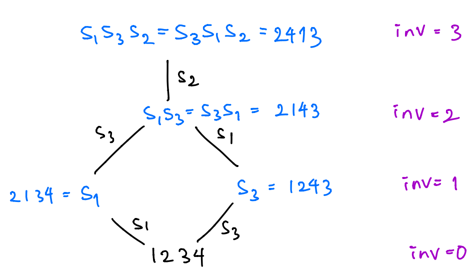
Facts

- **Theorem** $\ell(\pi) = \mathrm{inv}(\pi) :=$ number of inversions

- Multiplying by $(k, k+1)$ on the right corresponds to swapping π_k and π_{k+1}

Example Here $n = 3$, S_4, Coxeter group of type A_3.

$\pi = s_1 s_3 s_2 = s_3 s_1 s_2 = 2413$



The longest element, $\begin{pmatrix} 1 & 2 & \cdots & n, n+1 \\ n+1, n, \ldots & , 3, 2, 1 \end{pmatrix}$ often denoted w_0, has length $\binom{n+1}{2} = \frac{(n+1)(n)}{2}$.

- $w_0 \pi$ is the complement of π, Sage: w.complement()

$$\begin{pmatrix} 1 & 2 & 3 & 4 & \cdots & 3 & 2 & 1 \\ n+2-\pi_1 & n+2-\pi_2 & n+2-\pi_3 & n+2-\pi_4 & \cdots & n+2-\pi_{n-1} & n+2-\pi_n & n+2-\pi_{n+1} \end{pmatrix}$$

- πw_0 is the reverse of π, Sage: w.reverse()

$$\begin{pmatrix} 1 & 2 & 3 & \cdots & n-1, & n, & n+1 \\ \pi_{n+1} & \pi_n & \pi_{n-1} & \cdots & \pi_3 & \pi_2 & \pi_1 \end{pmatrix}$$

2.9 Coxeter element & upper/lower vertices

Def

Given $\pi \in S_{n+1}$, a <u>reduced word</u> for π is a way to write π as a shortest product of $\{s_1, s_2, \ldots, s_n\}$.

Note: each reduced word is a shortest path in the weak order of S_{n+1} from the identity permutation (minimum elt of the weak order) to π.

E.g. The permutation $1432 = s_3 s_2 s_3 = s_2 s_3 s_2$ has two reduced words.

Def A <u>Coxeter element</u> is a permutation $c \in S_{n+1}$ which can be written as a product of $\{s_1, s_2, s_3, \ldots, s_n\}$, each used exactly once.

E.g.
- $c = s_1 s_2 s_3 \ldots s_n$,

  $c = s_1 s_2 s_3 = (12)(23)(34) = (1234) = 2341 \in S_4$

  $c = s_1 s_2 s_3 s_4 s_5 s_6 = (12)(23)(34)(45)(56)(67) = (1234567) = 2345671 \in S_7$

- $c = \underbrace{\ldots s_5 s_3 s_1}_{\text{odd indices}} \underbrace{s_2 s_4 \ldots}_{\text{even indices}}$, $= s_1 s_3 s_5 \ldots s_2 s_4 \ldots$

  $c = s_1 s_3 s_2 = s_3 s_1 s_2 = (1243) = 2413 \in S_4$
  $(12)(34)(23) \quad (34)(12)(23)$

  $c = s_1 s_3 s_5 s_2 s_4 s_6 = (12)(34)(56)(23)(45)(67) = (1246753) = 2416375 \in S_7$
  $= s_5 s_3 s_1 s_2 s_4 s_6$

- $c = s_{\lceil \frac{n}{2} \rceil} \cdots s_1 \, s_{\lceil \frac{n}{2} \rceil + 1} \cdots s_n$

  $c = s_2 s_1 s_3$

  $c = s_3 s_2 s_1 s_4 s_5 s_6$
  $= s_3 s_2 s_4 s_1 s_5 s_6$

- A Coxeter elt $c = s_{i_1} s_{i_2} \cdots s_{i_n}$ written in cycle notation is of the form

$$(1\ d_1\ d_2 \cdots d_\ell,\ n{+}1,\ u_k\ u_{k-1} \cdots u_1) = (n{+}1,\ \overline{u_k,\ u_{k-1}, \cdots, u_1},\ 1,\ \underline{d_1, d_2, \ldots, d_\ell})$$

where $d_1 < d_2 \ldots < d_\ell$ are the lower-barred vertices of $\{2, \ldots, n\}$ and $u_1 < u_2 \ldots < u_k$ are the upper-barred vertices of $\{2, \ldots, n\}$.

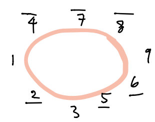
A polygon Q_c corresponding to c can be constructed as follows:

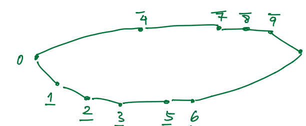
Place the numbers $1\ d_1\ d_2 \ldots d_\ell,\ n{+}1,\ u_k\ u_{k-1} \cdots u_1$ from this cycle around a circle.

E.g. $c = \overbrace{s_8 s_7 s_4}\ s_1\ \underbrace{s_2 s_3 s_5 s_6}$

$= (89)(78)(45)(12)(23)(34)(56)(67)$

$= (\underbrace{1\,2\,3\,5\,6}_{\text{lower-barred}}\,9\,\overbrace{8\,7\,4}^{\text{upper-barred}})$



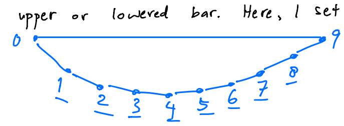


It won't make a difference whether 1 and n+1 are upper or lowered bar. Here, I set $\underline{1}$ and $\overline{9}$.

E.g. $c = \underbrace{s_1 s_2 s_3 s_4 s_5 s_6 s_7}$

$= (1\,\underbrace{2\,3\,4\,5\,6\,7}_{\substack{\text{must be}\\ \text{lowered bar}}}\,8)$

It won't make a difference whether 1 and n+1 are upper or lowered bar. Here, I set $\underline{1}$ and $\underline{8}$.

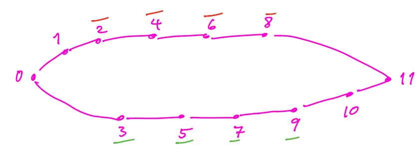


the Tamari case

E.g. $c = \overbrace{s_8 s_6 s_4 s_2}\ s_1\ \underbrace{s_3 s_5 s_7 s_9}$

$= (89)(67)(45)(23)(12)(34)(56)(78)(9,10)$

$= (\underbrace{1,3,5,7,9}_{\text{lower-barred}},10,\overbrace{8,6,4,2}^{\text{upper barred}})$

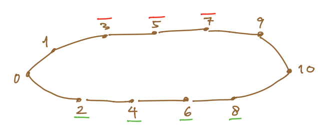


$c = \overbrace{s_3 s_5 s_7}\ s_1\ \underbrace{s_2 s_4 s_6 s_8} = s_3 s_5 s_1 s_7 s_2 s_4 s_6 s_8 = s_3 s_1 s_5 s_7 s_2 s_4 s_6 s_8 = s_1 s_3 s_5 s_7 s_2 s_4 s_6 s_8$

$= (34)(56)(78)(12)(23)(45)(67)(89)$

$= (\underbrace{1\,2\,4\,6\,8}_{\text{lower-barred}}\,9\,\overbrace{7\,5\,3}^{\text{upper-barred}})$

$= (9\,7\,5\,3\,1\,2\,4\,6\,8)$

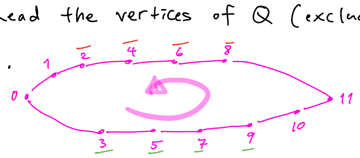


This polygon is the Q_{alt} that I've been using for the bipartite Cambrian lattice. If it's more natural to set $\underline{1}$, and set the barring of n+1 to match n, we can do that, too.

- To go from polygon Q to the Coxeter element c:

Read the vertices of Q (excluding 0, n+2) in counterclockwise order

E.g. Q=



$\mapsto c = (3\,5\,7\,9,\,10,\,8\,6\,4\,2\,1)$

$\boxed{\text{Fact}}$ Every Coxeter elt $c \in S_{n+1}$ has a reduced word of the form

$c = s_{u_k} s_{u_{k-1}} \cdots s_{u_1}\, s_1\, s_{d_1} s_{d_2} \cdots s_{d_\ell}$ where

$\overline{u_k} > \overline{u_{k-1}} > \ldots > \overline{u_1}$ which correspond to the upper vertices

$\underline{d_1} < \underline{d_2} < \cdots < \underline{d_\ell}$ correspond to the lower vertices

$\boxed{\text{PRACTICE}}$

For Coxeter element $c = s_{\lceil \frac{n}{2} \rceil} \cdots s_1\, s_{\lceil \frac{n}{2} \rceil + 1} \cdots s_n$, construct the polygon Q.

—— ended lecture Tue June 16 ——

2.10 c-sorting word

Def of c-sorting word

Given a Coxeter element $c \in S_{n+1}$, fix a reduced word $c = a_1 a_2 \cdots a_n$.
where $a_k \in \{s_1, \ldots, s_n\}$

Let $c^\infty := c|c|c|\cdots$
$= a_1 a_2 \cdots a_n | a_1 a_2 \cdots a_n | a_1 a_2 \cdots a_n | \cdots$

Given $\pi \in S_{n+1}$, the $(a_1, a_2, \ldots, a_n)$-sorting word for π is the subword of c^∞ which is lexicographically first (as a sequence of positions in c^∞) and is a reduced word for π.

- Every permutation has exactly one c-sorting word.
  The c-sorting word depends on the choice of reduced word $a_1 a_2 \ldots a_n$.

Example 1 Fix $c = s_1 s_3 s_2$. Note: this Coxeter elt c has two different reduced words. Here I use $s_1 s_3 s_2$.
$c^\infty = s_1 s_3 s_2 | s_1 s_3 s_2 | s_1 s_3 s_2 | s_1 s_3 s_2 | \cdots$

(i) $\pi = 1432 = s_3 s_2 s_3 = s_2 s_3 s_2$ has two reduced words.

Comparing
$s_1 s_3 s_2 | s_1 s_3 s_2 | s_1 s_3 s_2 | s_1 s_3 s_2 | \cdots$ and
$s_1 s_3 s_2 | s_1 s_3 s_2 | s_1 s_3 s_2 | s_1 s_3 s_2 | \cdots$,

the reduced word $s_3 s_2 s_3$ is lexicographically first (as a sequence of positions in c^∞)
so the c-sorting word for 1432 is $s_3 s_2 s_3$.

(ii) $w = 4132 = s_3 s_2 s_3 s_1 = s_2 s_3 s_2 s_1 = s_3 s_2 s_1 s_3$ has three reduced words.

Comparing
$s_1 s_3 s_2 | s_1 s_3 s_2 | s_1 s_3 s_2 | s_1 s_3 s_2 | \cdots$,
$s_1 s_3 s_2 | s_1 s_3 s_2 | s_1 s_3 s_2 | s_1 s_3 s_2 | \cdots$, and
$s_1 s_3 s_2 s_1 s_3 s_2 | s_1 s_3 s_2 | s_1 s_3 s_2 | \cdots$,

the reduced word $s_3 s_2 s_1 s_3$ is the c-sorting word for w.

Algorithm for finding the $(a_1, a_2, \ldots, a_n)$-sorting word of a permutation π, assuming we already know the length of π, and $\pi \neq \mathrm{Id}$. Ref: "Comb. Lattices & Beyond"

We write down a reduced word $\pi = u_1 u_2 \cdots u_\ell$ as follows, where $u_i \in S$.

- First, try each letter $a_1, a_2, \ldots, a_n$ (in this order) until we find one a_i s.t $\ell(a_i \pi) < \ell(\pi)$.
  Take a_i to be the first letter for π, set $u_1 := a_i$, and write $\pi' = u_1 \pi$.
- If $\pi' = \mathrm{Id}$, then $\ell = 1$ and u_1 is the desired $(a_1, a_2, \ldots, a_n)$-sorting word.

  Otherwise, try each of the n letters in the order $a_{i+1}, a_{i+2}, \ldots, a_n, a_1, \ldots$ until we find one $a_{i'}$ s.t $\ell(a_{i'} \pi') < \ell(\pi')$. Take $a_{i'}$ to be the second letter for π, set $u_2 := a_{i'}$, and define $\pi'' = u_2 \pi'$.

- If $\pi'' = \mathrm{Id}$, then $\ell = 2$ and $u_1 u_2$ is the desired $(a_1, a_2, \ldots, a_n)$-sorting word.
  Otherwise, continuing in this manner, we eventually find a reduced word for π.
  This is the $(a_1, a_2, \ldots, a_n)$-sorting word for π.

Example 2

Example of computing a c-sorting word (Ref: Thesis by Suleiman)

For $c = \overset{a_1}{s_1}\overset{a_2}{s_2}\overset{a_3}{s_3}\overset{a_4}{s_4}$

(i) Let $\pi = s_1 s_4 s_3 s_4 = (12)(35) = 21543$, and assume we know (for example by computing inv(π) with Sage) that length(π) = 4

$s_1\ s_2\ s_3\ s_4 \mid s_1\ s_2\ s_3\ s_4 \mid s_1\ s_2\ s_3\ s_4$

- $u_1 = s_1$ because $s_1 s_1 = Id$. Set $\pi' = u_1 \pi = s_1 s_1 s_4 s_3 s_4 = s_4 s_3 s_4$, of length 3
- Next, try each of $\underbrace{s_2, s_3, s_4, s_1}$, the next n letters in c^∞ after u_1,

$s_1\ \overbrace{s_2\ s_3\ s_4 \mid s_1}\ s_2\ s_3\ s_4 \mid s_1\ s_2\ s_3\ s_4$

until we find s_i s.t $\ell(s_i \pi') < \ell(\pi')$

* Try s_2: $s_2 \pi' = s_2 s_4 s_3 s_4$ has length 4 (To see this, we can check the number of inversions of π')
Doesn't work — keep trying

* Try s_3: $s_3 \pi' = s_3 s_4 s_3 s_4$
$= s_4 s_3 s_4 s_4$ by the long braid move
$= s_4 s_3$ since $s_4 s_4 = Id$
has length 2, smaller than $3 = \ell(\pi')$

So we set $u_2 = s_3$, and set $\pi'' = u_2 \pi'$
$= s_3 s_4 s_3 s_4$
$= s_4 s_3$

$s_1\ s_2\ s_3\ s_4 \mid s_1\ s_2\ s_3\ s_4 \mid s_1\ s_2\ s_3\ s_4$

- Next, try each of $\underbrace{s_4, s_1, s_2, s_3}$, the next n letters in c^∞ after u_2

$s_1\ s_2\ s_3\ \overbrace{s_4 \mid s_1\ s_2\ s_3}\ s_4 \mid s_1\ s_2\ s_3\ s_4$

until we find s_i s.t $\ell(s_i \pi'') < \ell(\pi'')$.

Then $u_3 = s_4$ because $s_4 \pi'' = s_4 s_4 s_3 = s_3$ has length 1. Set $\pi''' = s_3$

$s_1\ s_2\ s_3\ s_4 \mid s_1\ s_2\ s_3\ s_4 \mid s_1\ s_2\ s_3\ s_4$

- Next, $u_4 = s_3$ since $\pi''' = s_3$. ∴ The (s_1, s_2, s_3, s_4)-sorting word of π is $s_1 s_3 s_4 s_3$.

(ii) $\boxed{\text{PRACTICE}}$ Let $w = s_2 s_1 = (23)(12) = (132) = 31245$.
Verify this example
Following the algorithm, we get $s_2 s_1$ as the (s_1, s_2, s_3, s_4)-sorting word of w.

$s_1\ s_2\ s_3\ s_4 \mid s_1\ s_2\ s_3\ s_4 \mid s_1\ s_2\ s_3\ s_4$.

(iii) $\boxed{\text{PRACTICE}}$ Let $z = s_1 s_3 s_2 = s_3 s_1 s_2 = (1243) = 24135$.
Verify this example
Following the algorithm, we get $s_1 s_3 s_2$ as the (s_1, s_2, s_3, s_4)-sorting word of z.

$s_1\ s_2\ s_3\ s_4 \mid s_1\ s_2\ s_3\ s_4 \mid s_1\ s_2\ s_3\ s_4$.

2.11 c-sortable words

Def of c-sortable word: Let c be a Coxeter element, and $a_1 a_2 a_3 \cdots a_n$ be a reduced word of c.

For a set $K = \{ i_1 < i_2 < \cdots < i_r \} \subset [n]$, let

c_K denote $a_{i_1} a_{i_2} \cdots a_{i_r}$.

Ex $c = (a_1, a_2, a_3, a_4)$
$K = \{1, 2, 4\}$
$c_K = a_1 a_2 a_4$

The c-sorting word for a permutation $\pi \in S_{n+1}$ can be uniquely written as

$\pi = c_{K_1} c_{K_2} \cdots c_{K_p}$, where p is minimal.

If $K_1 \supset K_2 \supset \cdots \supset K_p$, we say that

π is c-sortable and $c_{K_1} c_{K_2} \cdots c_{K_p}$ is a c-sortable word.

• Not every permutation is c-sortable.

Fact: Although the def of c-sortable requires a choice of a reduced word of c, the set of c-sortable elements does not depend on the choice of reduced word for c.

From above Example 1,

for $c = s_1 s_3 s_2 = s_3 s_1 s_2$

c has two reduced words – here I choose $s_1 s_3 s_2$. But whether or not a permutation is c-sortable does not depend on my choice of reduced words.

(i) $\pi = 1432 = s_3 s_2 s_3 = s_1 s_3 s_2 \mid s_1 s_3 s_2 \mid s_1 s_3 s_2 \mid s_1 s_3 s_2 \mid \cdots$

$\{s_3, s_2\} \supset \{s_3\} \supset \emptyset \ \cdots$

is the c-sorting word for π, so π is c-sortable.

(ii) $w = 4132 = s_3 s_2 s_1 s_3 = s_1 s_3 s_2 \mid s_1 s_3 s_2 \mid s_1 s_3 s_2 \mid s_1 s_3 s_2 \mid \cdots$

$\{s_3, s_2\} \not\supseteq \{s_1\}$

is the c-sorting word for w, so w is not c-sortable.

From above example 2, for $c = s_1 s_2 s_3 s_4$

(i) $\pi = s_1 s_3 s_4 s_3$ $\quad s_1 s_2 s_3 s_4 \mid s_1 s_2 s_3 s_4 \mid s_1 s_2 s_3 s_4$ is c-sortable.

$\{s_1, s_3, s_4\} \supseteq \{s_3\} \supseteq \emptyset \ \cdots$

(ii) $w = s_2 s_1$ $\quad s_1 s_2 s_3 s_4 \mid s_1 s_2 s_3 s_4 \mid s_1 s_2 s_3 s_4$

$\{s_2\} \not\supseteq \{s_1\}$

(iii) $z = s_1 s_3 s_2$ $\quad s_1 s_2 s_3 s_4 \mid s_1 s_2 s_3 s_4 \mid s_1 s_2 s_3 s_4$

$\{s_1, s_3\} \not\supseteq \{s_2\}$

(ii), (iii): not c-sortable

$\boxed{\text{Thm 2.11}}$ Let c be any Coxeter elt.

- The identity permutation is c-sortable (since it is the empty word).
- The longest elt w_0 is c-sortable.

Example 3 for Thm 2.11

Apply the algorithm 2.10 for A_3, $w_0 = 4321$. We know w_0 has length $\binom{4}{2} = 6$.

For $c = s_1 s_2 s_3$ (Tamari Coxeter element)

Note: Multiplying $s_i = (i, i+1)$ on the left corresponds to swapping values $i, i+1$

First, apply s_1: $\quad s_1 w_0 = s_1 \circ 4321 \overset{\substack{\text{swaps}\\2,1}}{=} 4312$.

Set $u_1 = s_1$, and $w_0' := s_1 w_0 = 4312$

$s_1\ s_2\ s_3 \mid s_1\ s_2\ s_3 \mid s_1\ s_2\ s_3$

Try the next letter in c^∞, which is s_2: $\quad s_2 w_0' = s_2 \circ 4312 \overset{\substack{\text{swaps}\\3,2}}{=} 4213$

This puts 2,3 in order, which decreases the inversion number,

so $\ell(s_2 w_0') < \ell(w_0')$.

Set $u_2 = s_2$, and $w_0'' := s_2 w_1' = 4213$

$s_1\ s_2\ s_3 \mid s_1\ s_2\ s_3 \mid s_1\ s_2\ s_3$

Try the next letter in c^∞, which is s_3: $\quad s_3 w_0'' = s_3 \circ 4213 \overset{\substack{\text{swaps}\\4,3}}{=} 3214$

Again, $\ell(s_3 \circ w_0'') < \ell(w_0'')$.

Set $u_3 = s_3$, and $w_0''' := s_3 w_0'' = 3214$

$s_1\ s_2\ s_3 \mid s_1\ s_2\ s_3 \mid s_1\ s_2\ s_3$

The next letter in c^∞, s_1, also works because entries 2,1 in $w_0''' = 3\underbrace{21}4$ are out of order.

Set $u_4 = s_1$, and $w_0^{(4)} := s_1 w_0''' = 3124$.

The next letter in c^∞, s_2, also works because entries 3,2 in $w_0^{(4)} = \underbrace{3\,1\,2}4$ are out of order.

Set $u_5 = s_2$, and $w_0^{(5)} := s_2 w_0^{(4)} = 2134$.

$s_1\ s_2\ s_3 \mid s_1\ s_2\ s_3 \mid s_1\ s_2\ s_3$

The next letter in c^∞, s_3, does not work because $s_3 w_0^{(5)} = s_3 \circ 21\underbrace{34}_{\substack{3,4\\ \text{in}\\ \text{order}}} = 21\underbrace{43}_{\substack{3,4\\ \text{out of}\\ \text{order}}}$

→ The next letter in c^∞ after s_3 is s_1:

s_1 works because $s_1 \circ w_0^{(5)} = s_1 \circ 2134 \overset{\substack{\text{swaps}\\2,1}}{=} 1234$.

Set $u_6 = s_1$

$s_1\ s_2\ s_3 \mid s_1\ s_2\ s_3 \mid s_1\ s_2\ s_3$

Note: $w_0 = s_1 s_2 s_3 s_2 s_1 s_2$ is not the (s_1, s_2, s_3)-sorting word

∴ The (s_1, s_2, s_3)-sorting word for $w_0 = 4321$ is $s_1 s_2 s_3 s_1 s_2 s_1$.

Since $\{s_1, s_2, s_3\} \supset \{s_1, s_2\} \supset \{s_1\}$, this shows $w_0 = 4321$ is c-sortable.

Example for Thm 2.11

Apply the algorithm 2.10 for A_4, $w_0 = 54321$. We know w_0 has length $\binom{5}{2} = 10$.
For $c = s_3 s_1 s_2 s_4$ (a "bipartite" Coxeter element)

Note: Multiplying $s_i = (i, i+1)$ on the left corresponds to swapping values $i, i+1$

First, try s_3: $s_3 w_0 = s_3 \circ 54321 = 53421$.
Set $u_1 = s_3$, and $w_0' := s_3 w_0 = 53421$

$s_3\ s_1\ s_2\ s_4 \mid s_3\ s_1\ s_2\ s_4 \mid s_3\ s_1\ s_2\ s_4$

Try the next letter in c^∞, which is s_1 : $s_1 w_0' = s_1 \circ 53421 \overset{\text{swap } 1,2}{=} 53412$
This puts 1, 2 in order, which decreases the inversion number,
so $\ell(s_1 w_0') < \ell(w_0')$.
Set $u_2 = s_1$, and $w_0'' := s_1 w_0' = 53412$.

$s_3\ s_1\ s_2\ s_4 \mid s_3\ s_1\ s_2\ s_4 \mid s_3\ s_1\ s_2\ s_4$

The next letter in c^∞, s_2, works: $s_2 w_0'' = s_2 \circ 53412 \overset{\text{swaps } 2,3}{=} 52413$
Set $u_3 = s_2$, and $w_0''' := s_2 w_0'' = 52413$.

Next, set
$u_4 = s_4$, and $w_0^{(4)} := s_4 w_0''' = s_4 \circ 52413 = 42513$
$u_5 = s_3$, and $w_0^{(5)} := s_3 w_0^{(4)} = s_3 \circ 42513 = 32514$
$u_6 = s_1$, and $w_0^{(6)} := s_1 w_0^{(5)} = s_1 \circ 32514 = 31524$
$u_7 = s_2$, and $w_0^{(7)} := s_2 w_0^{(6)} = s_2 \circ 31524 = 21534$
$u_8 = s_4$, and $w_0^{(8)} := s_4 w_0^{(7)} = s_4 \circ 21534 = 21435$
$u_9 = s_3$, and $w_0^{(9)} := s_3 w_0^{(8)} = s_3 \circ 21435 = 21345$
$u_{10} = s_1$, and $w_0^{(10)} := s_1 w_0^{(9)} = s_1 \circ 21345 = 12345$

PRACTICE
Verify computation for $u_4, \dots, u_{10}$

$s_3\ s_1\ s_2\ s_4 \mid s_3\ s_1\ s_2\ s_4 \mid s_3\ s_1\ s_2\ s_4$, So the (s_3, s_1, s_2, s_4)-sorting word for 54321 is $s_3 s_1 s_2 s_4 s_3 s_1 s_2 s_4 s_3 s_1$, and so 54321 is $s_3 s_1 s_2 s_4$-sortable.

PRACTICE
Use Algorithm 2.10 to verify these two examples

More examples for type A_4 Coxeter group

- If $c = s_1 s_3 s_5 s_2 s_4$, $w_0 = s_1 s_3 s_5 s_2 s_4 \mid s_1 s_3 s_5 s_2 s_4 \mid s_1 s_3 s_5 s_2 s_4$ is the c-sorting word of w_0, which shows w_0 is c-sortable.
- If $c = s_1 s_2 s_3 s_4 s_5$, $w_0 = s_1 s_2 s_3 s_4 s_5 \mid s_1 s_2 s_3 s_4\, s_5 \mid s_1\ 2\ s_3\, s_4 s_5 \mid s_1 s_2\, s_3 s_4 s_5 \mid s_1\, s_2 s_3 s_4 s_5$ is the c-sorting word of w_0, which shows w_0 is c-sortable.

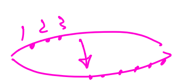
REU Exercise 11

Exercises 8 & 9 give us the c-sorting word (of w_0) for (Tamari) $c = s_1 s_2 \dots s_n$ and (bipartite) $c = $ (odd indices) (even indices)

Now, consider the Coxeter element $c := \underbrace{s_{\lceil \frac{n}{2} \rceil} \dots s_3 s_2 s_1}_{\text{the first half of indices}}\ \underbrace{s_{\lceil \frac{n}{2} \rceil + 1}, \dots, s_{n-2}, s_{n-1}, s_n}_{\text{the second half of indices}}$

for example
$c = s_2 s_1 s_3$
$c = s_2 s_1 s_3 s_4$
$c = s_3 s_2 s_1 s_4 s_5$
$c = s_3 s_2 s_1 s_4 s_5 s_6$
$c = s_4 s_3 s_2 s_1 s_5 s_6 s_7$

"Halfway"



Give the c-sorting word of w_0 for c for arbitrary n.

<u>Known</u> For Tamari, $\lambda_1 - 4 = \lambda_2$ for $n \geqslant 4$, for $c = s_1 s_2 \dots s_n$

$\lambda_1 - 2 = \lambda_2$ for $n \geqslant 4$, for $c = s_1 s_3 s_5 \dots s_2 s_4 \dots$

<u>Conjecture</u> For "halfway" c, $\lambda_1 - 3 = \lambda_2$ for $n \geqslant 4$.

<u>Thm 2.11 (b)</u>

A permutation π is the minimum elt of its η_c-fiber iff π is c-sortable.

~ end Thu June 18, 2020 ~ jg 10/10