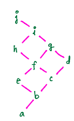
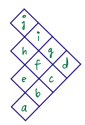
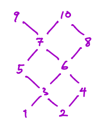
References

- Serge Elnitsky "Rhombic tilings of polygons and classes of reduced words in Coxeter groups" 1994

- Robert Bédard "On commutation classes of reduced words in Weyl groups" 1999

for graphs like  or  and 

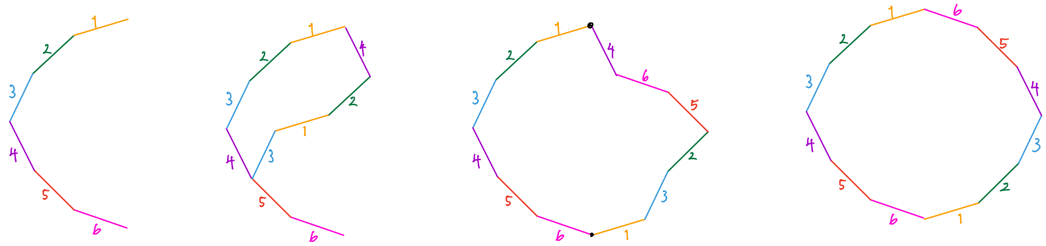
Ordered Rhombic tilings of Elnitsky polygons
&
Linear extensions of heaps

Elnitsky polygon $E(\mathrm{Id})$

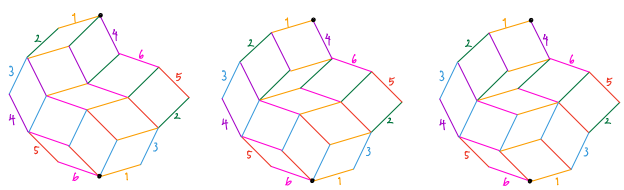
Elnitsky polygon $E(421356)$

Elnitsky polygon $E(465231)$

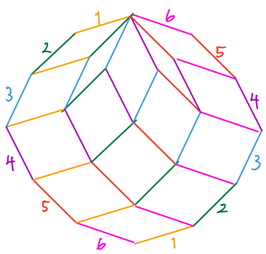
Elnitsky polygon $E(654321)$



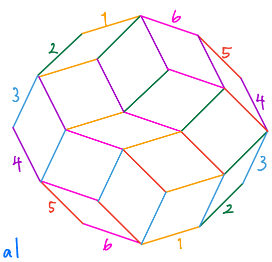
A few Rhombic tilings of $E(465231)$:



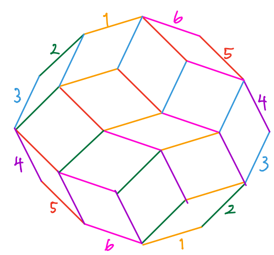
A few tilings of $E(654321)$:



Tamari
$c = s_1 s_2 s_3 s_4 s_5$



Our usual bipartite with
$c = (\text{odd } s_i)(\text{even } s_i) = s_1 s_3 s_5 s_2 s_4$



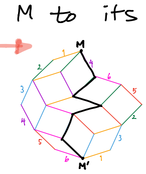
Bipartite but with
$c = (\text{even } s_i)(\text{odd } s_i) = s_2 s_4 s_1 s_3 s_5$

<u>Thm</u> (Elnitsky 94) Suppose $\pi \in S_n$.

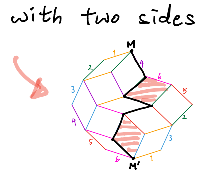
- $\{$The rhombic tilings of the Elnitsky polygon $E(\pi)\}$ $\overset{\text{bijection}}{\longleftrightarrow}$ $\{$The commutation classes of $\pi\}$.
- The number of tiles in a tiling of $E(\pi)$ depends on π and is equal to $\mathrm{inv}(\pi) = \mathrm{length}(\pi)$.

<u>Def</u> Let $\pi \in S_n$.

- Given a tiling of the Elnitsky polygon $E(\pi)$, a path joining M to its antipode M' and consisting of precisely n tile edges is called a <u>border</u>

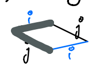
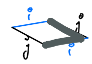


- <u>Fact</u> Any border (except the rightmost) has at least one tile (located to the right of it) which touches it with two sides

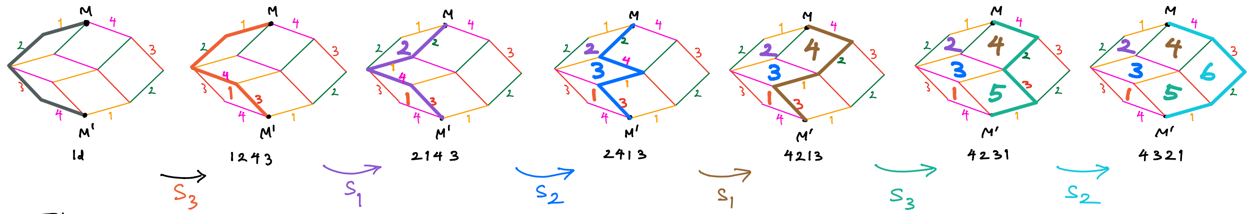


Def/fact

Given a tiling of the Elnitsky polygon $E(\pi)$, we can order the tiles by labeling them with numbers $1, 2, \dots, \ell$, where $\ell = \text{length}(\pi)$, as follows:

- Take the leftmost border, assign "1" to some tile which touches the border w/ 2 sides
- Replace, in the border  with 
- Assign "2" to some tile which touches the border w/ 2 sides, and so on.

Example $E(4321)$



Thm

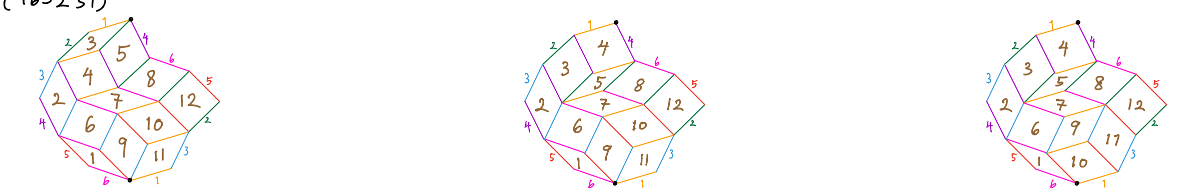
Given a tiling T of the Elnitsky polygon $E(\pi)$,

$\{$reduced words in the commutation class corresponding to $T\}$ $\xleftrightarrow{\text{bijection}}$ $\{$the orderings of the tiles in the tiling $T\}$

Given a reduced word $u_1u_2 \dots u_\ell$ of π, start w/ the leftmost border of polygon $E(\pi)$, then add the tile corresponding to u_1, label it "1". Add the tile corresponding to u_2, label the tile "2", and so on.

Example of various ordered tilings $\longleftrightarrow$ reduced words of 465231

$E(465231)$



$s_5\ s_3\ s_1\ s_2\ s_1\ s_4\ s_3\ s_2\ s_5\ s_4\ s_5\ s_3$
$u_1\ u_2\ u_3\ u_4\ u_5\ u_6\ u_7\ u_8\ u_9\ u_{10}\ u_{11}\ u_{12}$

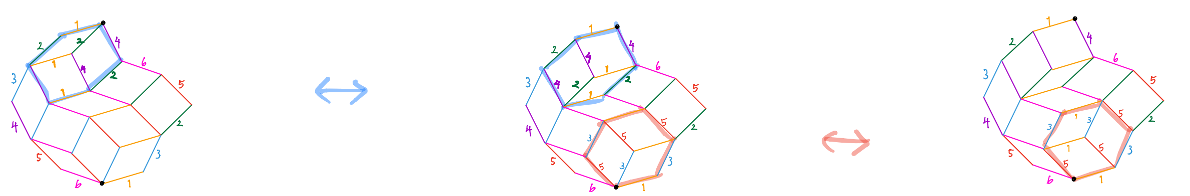
$s_5\ s_3\ s_2\ s_1\ s_2\ s_4\ s_3\ s_2\ s_5\ s_4\ s_5\ s_3$
$u_1\ u_2\ u_3\ u_4\ u_5\ u_6\ u_7\ u_8\ u_9\ u_{10}\ u_{11}\ u_{12}$

$s_5\ s_3\ s_2\ s_1\ s_2\ s_4\ s_3\ s_2\ s_4\ s_5\ s_4\ s_3$
$u_1\ u_2\ u_3\ u_4\ u_5\ u_6\ u_7\ u_8\ u_9\ u_{10}\ u_{11}\ u_{12}$

Prop

Two tilings of $E(\pi)$ are connected by a sequence of hexagon flips

$E(465231)$

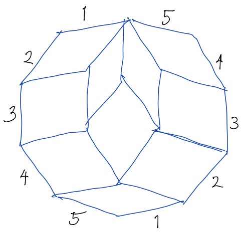


$s_5\ s_3\ s_1\ s_2\ s_1\ s_4\ s_3\ s_2\ s_5\ s_4\ s_5\ s_3$
$u_1\ u_2\ u_3\ u_4\ u_5\ u_6\ u_7\ u_8\ u_9\ u_{10}\ u_{11}\ u_{12}$

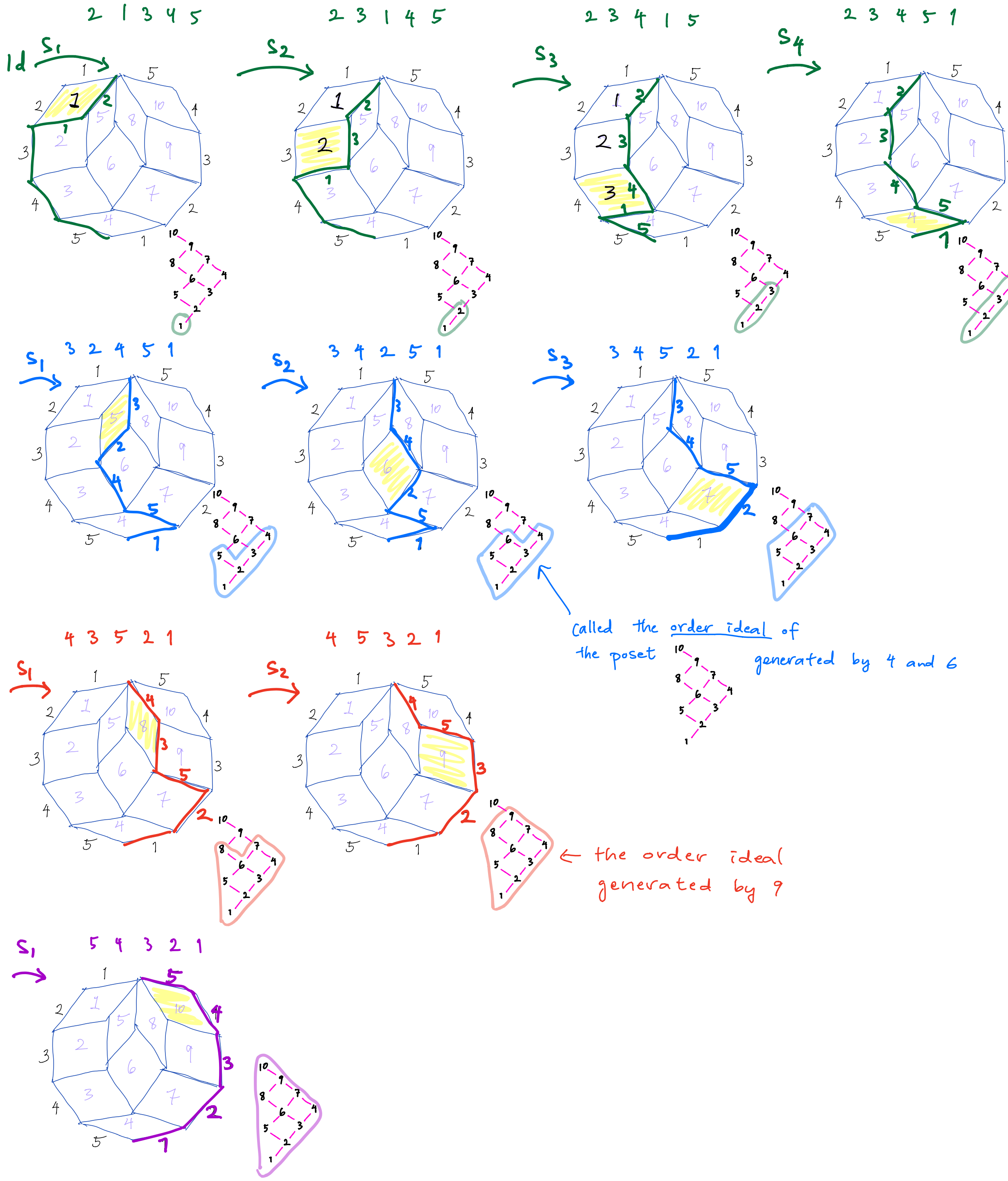
$s_5\ s_3\ s_2\ s_1\ s_2\ s_4\ s_3\ s_2\ s_5\ s_4\ s_5\ s_3$
$u_1\ u_2\ u_3\ u_4\ u_5\ u_6\ u_7\ u_8\ u_9\ u_{10}\ u_{11}\ u_{12}$

$s_5\ s_3\ s_2\ s_1\ s_2\ s_4\ s_3\ s_2\ s_4\ s_5\ s_4\ s_3$
$u_1\ u_2\ u_3\ u_4\ u_5\ u_6\ u_7\ u_8\ u_9\ u_{10}\ u_{11}\ u_{12}$

The tiling of the Elnitsky polygon
for $w_0 = 54321$
corresponding to the commutation class
containing the c-sorting word of w_0
for $c = s_1 s_2 s_3 s_4$
(Tamari Coxeter element)



$w_0 = s_1 s_2 s_3 s_4$
$s_1 s_2 s_3$
$s_1 s_2$
s_1



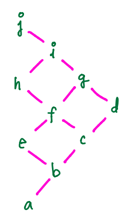
Def Let $(P, \leq)$ be a poset w/ ℓ elements

A linear extension f of P is a total ordering of the elements of P which respects the partial order $\leq$.

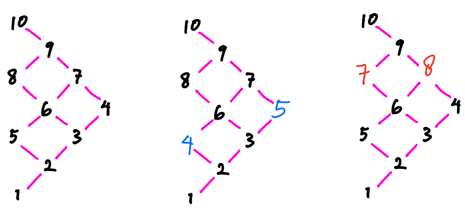
That is, f is a bijection $f: P \to [\ell]$ so that

$$x \leq_P y \implies f(x) \underset{\text{usual integer ordering}}{\leq} f(y)$$

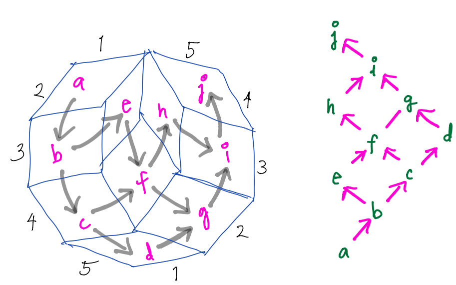
Example

Hasse diagram of poset 

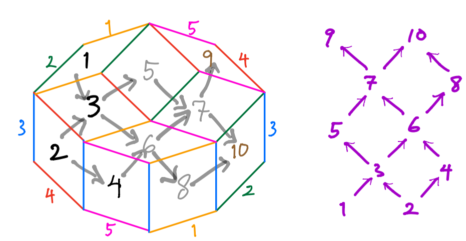
This poset has twelve linear extensions, for example
(see all twelve on the next page)



You can associate a tiling with the Hasse diagram of a poset, for example



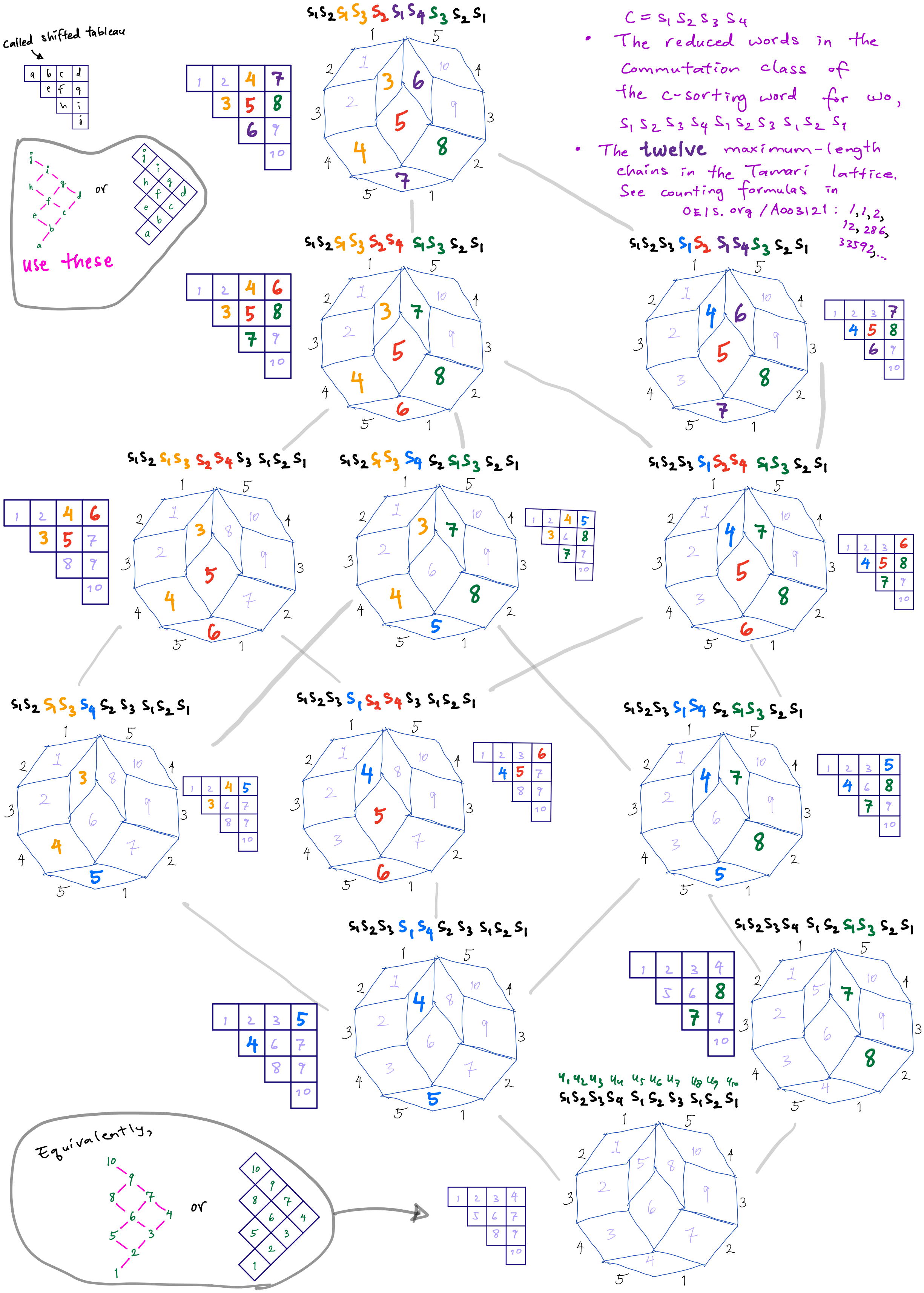
Tamari



Our usual bipartite
$c =$ (odd indices) (even indices)

$c = s_1 s_2 s_3 s_4$

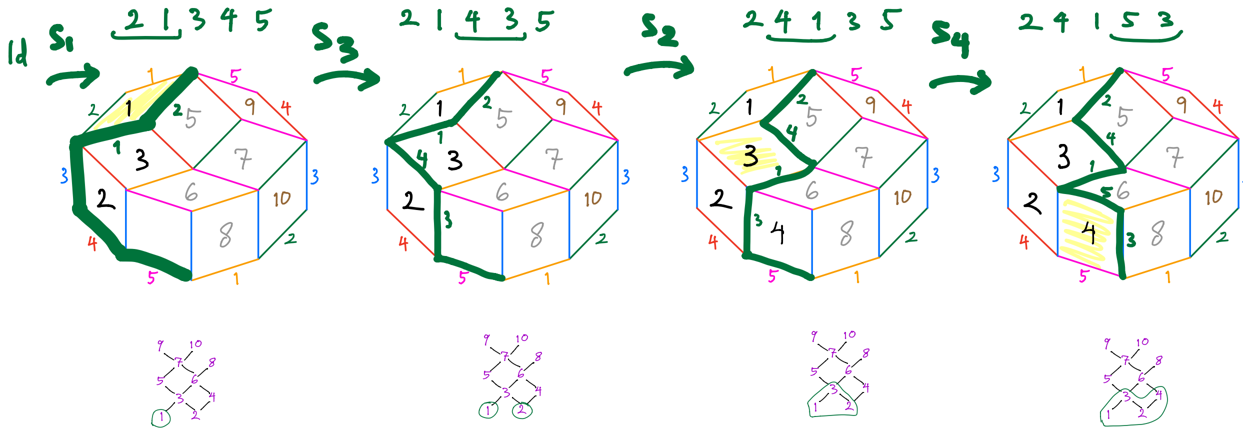
- The reduced words in the commutation class of the c-sorting word for w_0, $s_1 s_2 s_3 s_4 s_1 s_2 s_3 s_1 s_2 s_1$
- The **twelve** maximum-length chains in the Tamari lattice. See counting formulas in OEIS.org/A003121: 1, 1, 2, 12, 286, 33592, ...

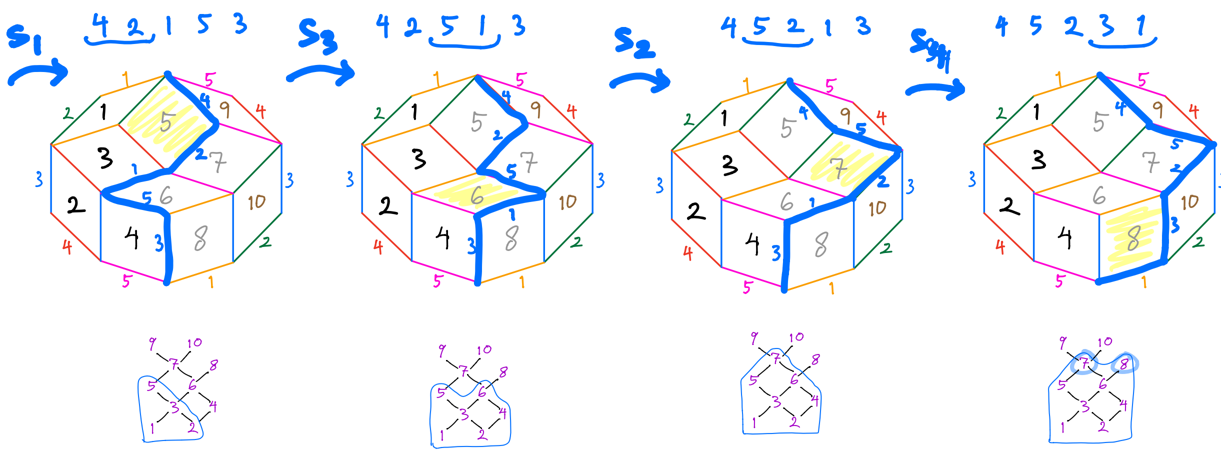


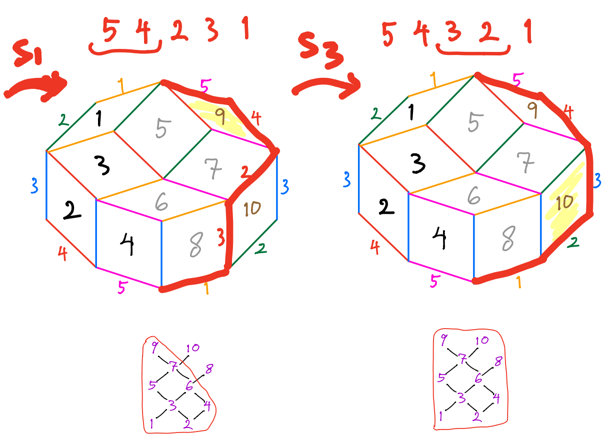
The tiling of the Elnitsky polygon for $w_0 = 54321$

corresponding to the commutation class containing the c-sorting word of w_0

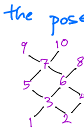
for $c = s_1 s_3 s_2 s_4$ (our usual bipartite Coxeter element)







called the order ideal of the poset

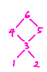
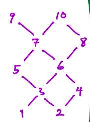
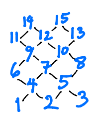
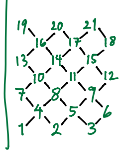


generated by 7 and 8

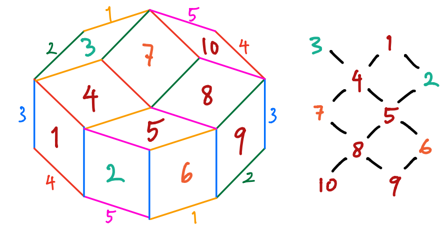
$c = s_1 s_3 s_2 s_4$

- The reduced words in the commutation class of the c-sorting word for w_0, $s_1 s_3 s_2 s_4 s_1 s_3 s_2 s_4 s_1 s_3$

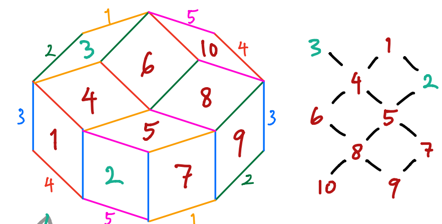
- There are seventy reduced words in the commutation class of this c-sorting word. I only drew several of them. (Not in OEIS)

n	count (using Sage)	heap	c-sorting word of w_0
2	1		$s_1 s_2 s_1$
3	4		$s_1 s_3 s_2 s_1 s_3 s_2$
4	70		$s_1 s_3 s_2 s_4$ $s_1 s_3 s_2 s_4$ $s_1 s_3$
5	7896		$s_1 s_3 s_5 s_2 s_4$ $s_1 s_3 s_5 s_2 s_4$ $s_1 s_3 s_5 s_2 s_4$
6	6648040		$s_1 s_3 s_5 s_2 s_4 s_6$ $s_1 s_3 s_5 s_2 s_4 s_6$ $s_1 s_3 s_5 s_2 s_4 s_6$ $s_1 s_3 s_5$

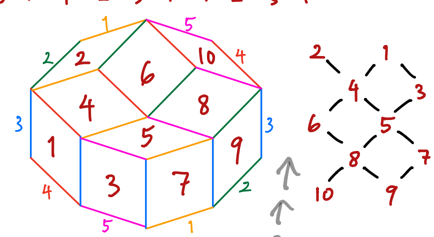
$s_3 s_4 s_1 s_2 s_3 s_4 s_1 s_2 s_3 s_1$



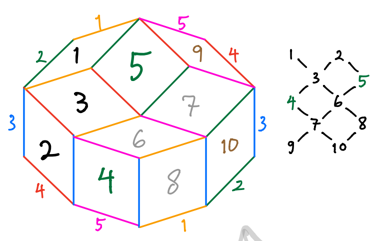
$s_3 s_4 s_1 s_2 s_3 s_1 s_4 s_2 s_3 s_1$



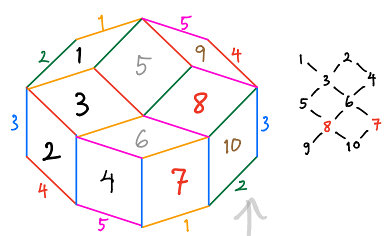
$s_3 s_1 s_4 s_2 s_3 s_1 s_4 s_2 s_3 s_1$



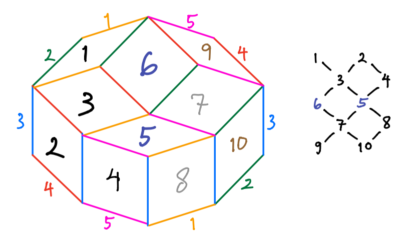
$s_1 s_3 s_2 s_1 s_4 s_3 s_2 s_4 s_1 s_3$



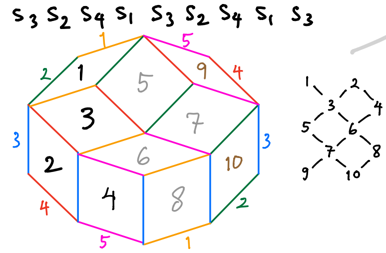
$s_1 s_3 s_2 s_4 s_1 s_3 s_4 s_2 s_1 s_3$



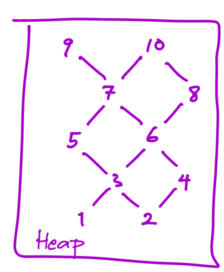
$s_1 s_3 s_2 s_4 s_3 s_1 s_2 s_4 s_1 s_3$



$s_1 s_3 s_2 s_4 s_1 s_3 s_2 s_4 s_1 s_3$



or

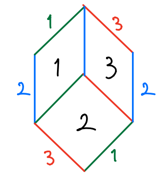
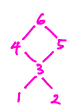
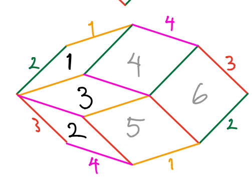
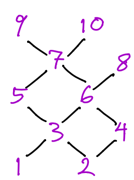
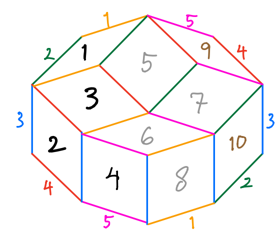
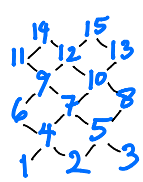
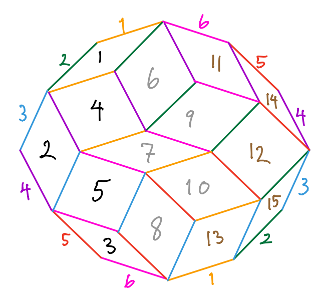
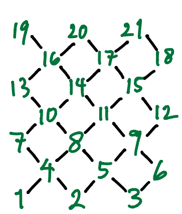
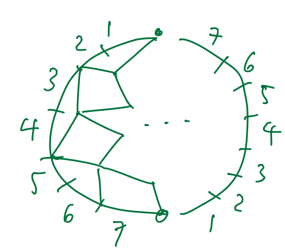


use this convention

There are seventy reduced words in the commutation class of the c-sorting word of w_0 for $c = s_1 s_3 s_2 s_4$. Here $n=4$. (Not in OEIS)

Thm (maybe Stembridge) $\{$reduced words in a commutation class ζ of $w_0\}$ $\overset{1-1}{\longleftrightarrow}$ $\{$linear extensions of the heap corresponding to $\zeta\}$.

Data for other n.

n	count (using Sage)	heap	c-sorting word of w_0	(ordered) tiling
2	1		$s_1 s_2 s_1$	
3	4		$s_1 s_3 s_2 s_1 s_3 s_2$	
4	70		$s_1 s_3 s_2 s_4$ $s_1 s_3 s_2 s_4$ $s_1 s_3$	
5	7896		$s_1 s_3 s_5 s_2 s_4$ $s_1 s_3 s_5 s_2 s_4$ $s_1 s_3 s_5 s_2 s_4$	
6	6648040		$s_1 s_3 s_5 s_2 s_4 s_6$ $s_1 s_3 s_5 s_2 s_4 s_6$ $s_1 s_3 s_5 s_2 s_4 s_6$ $s_1 s_3 s_5$	

The tiling of the Elnitsky polygon for $w_0 = 654321$

corresponding to the commutation class containing the c-sorting word of w_0

for $c = s_1 s_3 s_5 s_2 s_4$ (our usual bipartite Coxeter element)