Overview only given

Thurs, June 11, 2020

Lectures:
Fri, June 12, 2020
Mon, June 15, 2020

Notes for PROBLEM II (part 2)

Cambrian Lattices

Ref: papers by Nathan Reading

(Last update: Tue June 16)

2.4 (Type A) Cambrian lattices

We generalize the map η_{tamari}. Partition the set $[n+1] = \{1, 2, \ldots, n+1\}$ into two sets, upper-barred set $\overline{[n+1]}$ and lower-barred set $\underline{[n+1]}$

$$\text{Eg. } \underline{[6]} = \{4,6\}, \ \overline{[6]} = \{1,2,3,5\}$$

Let Q be a polygon w/ vertices $0, \ldots, n+2$ drawn from left to right

Draw 0 and $n+2$ on the same horizontal line.
Put upper-barred vertices above this horizontal line.
Put lower-barred vertices below this horizontal line.

Polygon Q =

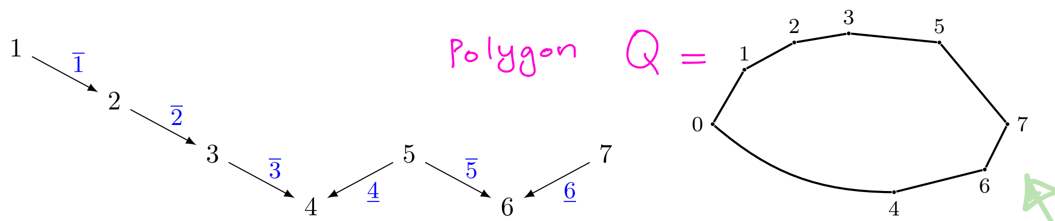

FIGURE 1. The polygon $P(Q)$ and quiver Q with $\underline{[6]} = \{4, 6\}$ and $\overline{[6]} = \{1, 2, 3, 5\}$.

First, use Q to partition the set $[n+1]$ into two sets, the *upper-barred* integers $\overline{[n+1]}$ and the *lower-barred* integers $\underline{[n+1]}$, as follows. Let $\overline{[n+1]}$ be the set of all vertices i such that $i \to (i+1)$ is in Q and let $\underline{[n+1]}$ be the set of all vertices i such that $i \leftarrow (i+1)$ is in Q.

Next, associate to Q an $(n+3)$-gon $P(Q)$ with vertex labels $0, 1, 2, \ldots, n+2$. Draw the vertices $0, 1, \ldots, n+2$ in order from left to right so that: (1) the vertices 0 and $n+2$ are placed on the same horizontal line L; (2) the upper-barred vertices are placed above L; (3) the lower-barred vertices are placed below L. See Figure 1.

Given a permutation $\pi \in S_{n+1}$, we write π in one-line notation as $\pi_1 \pi_2 \ldots \pi_{n+1}$, where $\pi_i = \pi(i)$ for $i \in [n+1]$. For each $i \in \{0, \ldots, n+1\}$, define $\lambda_i(\pi)$ to be a path from the left-most vertex 0 to the right-most vertex $n+2$ as follows. Let $\lambda_0(\pi)$ be the path from the vertex 0 to the vertex $n+2$ passing through all lower-barred vertices $\underline{i} \in \underline{[n+1]}$ in numerical order. Thus $\lambda_0(\pi)$ is the path along the lower boundary edges of $P(Q)$. Define $\lambda_1(\pi)$ as the piecewise linear path from 0 to $n+2$ passing through the vertices

polygon Q

$$\begin{cases} \underline{[n+1]} \cup \{\pi_1\}, & \text{if } \pi_1 \in \overline{[n+1]}; \\ \underline{[n+1]} \setminus \{\pi_1\}, & \text{if } \pi_1 \in \underline{[n+1]}, \end{cases}$$

maintaining the numerical order of the vertices visited. Repeating this process recursively, the final path $\lambda_{n+1}(\pi)$ passes from 0 to $n+2$ through all upper-barred vertices $\bar{i} \in \overline{[n+1]}$. Thus $\lambda_{n+1}(\pi)$ is the path along the upper boundary edges of $P(Q)$.

polygon Q

Definition 2.1. [Rea06] Define a map $\eta_Q \colon S_{n+1} \to \{\text{triangulations of } P(Q)\}$, $\pi \mapsto \eta_Q(\pi)$, where $\eta_Q(\pi)$ is the triangulation (including the boundary edges) of $P(Q)$ that arises as the union of the paths $\lambda_0(\pi), \ldots, \lambda_{n+1}(\pi)$.

Remark 2.2. It is shown in [Rea06] that η_Q is surjective and that its fibers correspond to the congruence classes of a certain lattice congruence on the weak order on the symmetric group. The induced poset structure is a lattice called a *Cambrian lattice* of type $\mathbb{A}$. More precisely, two triangulations are ordered $\mathcal{T} \leq \mathcal{T}'$ if there exist permutations $\pi \leq \pi'$ in the weak order such that $\eta_Q(\pi) = \mathcal{T}$ and $\eta_Q(\pi') = \mathcal{T}'$.

Example 2.3. Let Q be the quiver in Figure 1, where $n = 5$. Then $\overline{[6]} = \{\bar{1}, \bar{2}, \bar{3}, \bar{5}\}$ and $\underline{[6]} = \{\underline{4}, \underline{6}\}$. Let $\pi = 4\,5\,3\,1\,2\,6 \in S_6$, written in one-line notation.

Then the paths $\lambda_i(\pi)$ described above are as follows.

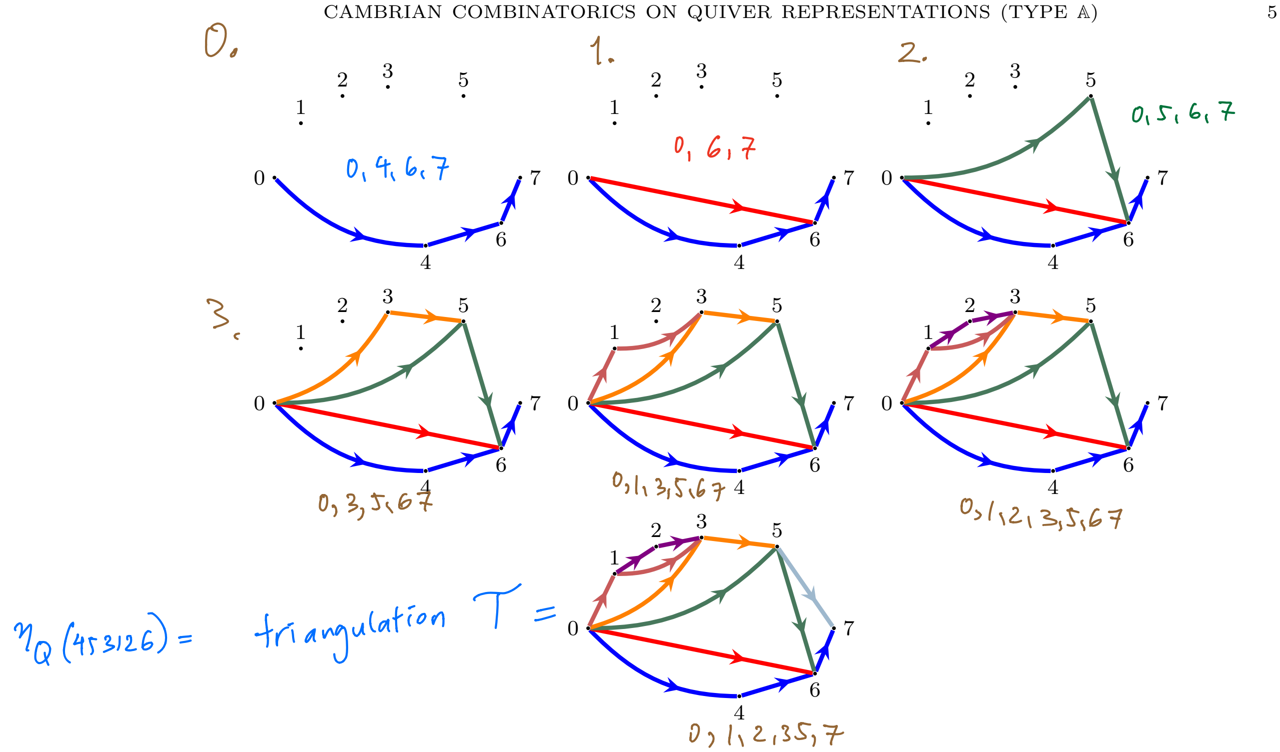

FIGURE 2. The paths $\lambda_i(453126)$ and triangulation for $\eta_Q(453126)$ from Example 2.3

$\lambda_0(\pi) = 0, \underline{4}, \underline{6}, 7$	
$\lambda_1(\pi) = 0, \underline{6}, 7$	delete $\pi(1) = \underline{4}$ from $\lambda_0(\pi)$
$\lambda_2(\pi) = 0, \overline{5}, \underline{6}, 7$	add $\pi(2) = \overline{5}$ to $\lambda_1(\pi)$
$\lambda_3(\pi) = 0, \overline{3}, \overline{5}, \underline{6}, 7$	add $\pi(3) = \overline{3}$ to $\lambda_2(\pi)$
$\lambda_4(\pi) = 0, \overline{1}, \overline{3}, \overline{5}, \underline{6}, 7$	add $\pi(4) = \overline{1}$ to $\lambda_3(\pi)$
$\lambda_5(\pi) = 0, \overline{1}, \overline{2}, \overline{3}, \overline{5}, \underline{6}, 7$	add $\pi(5) = \overline{2}$ to $\lambda_4(\pi)$
$\lambda_6(\pi) = 0, \overline{1}, \overline{2}, \overline{3}, \overline{5}, 7$	delete $\pi(6) = \underline{6}$ from $\lambda_5(\pi)$

$\eta^{-1}(\mathcal{T})$ = 456312, 453612, 453162, 453126; $s_3 = (3,4)$, $s_4 = (4,5)$, $s_5 = (5,6)$

The triangulation $\mathcal{T} = \eta_Q(\pi)$ is given in Figure 2. Note that the fiber of $\mathcal{T}$ is

sage: Permutation([...]).inversions()

$$\eta_Q^{-1}(\mathcal{T}) = \{\underset{\text{inv}=8}{453126}, \underset{\text{inv}=9}{453162}, \underset{\text{inv}=10}{453612}, \underset{\text{inv}=11}{456312}\}.$$

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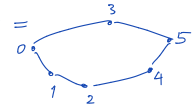
3. REPRESENTATIONS OF QUIVERS OF TYPE $\mathbb{A}$

Let $\Bbbk$ be an algebraically closed field, for example, $\Bbbk = \mathbb{C}$. Given a quiver Q, we denote by Q_0 the set of its vertices and by Q_1 its set of arrows. For $\alpha \in Q_1$, let $s(\alpha)$ be the source of α and $t(\alpha)$ be its target. A *path* from i to j in Q is a sequence of arrows $\alpha_1\alpha_2 \dots \alpha_\ell$ such that $s(\alpha_1) = i$, $t(\alpha_\ell) = j$, and $t(\alpha_h) = s(\alpha_{h+1})$, for all $1 \le h \le \ell - 1$. The integer ℓ is called the *length* of the path. Paths of length zero are called *constant paths* and are denoted by e_i, $i \in Q_0$.

A *representation* $M = (M_i, \varphi_\alpha)$ of Q consists of a $\Bbbk$-vector space M_i, for each vertex $i \in Q_0$, and a $\Bbbk$-linear map $\varphi_\alpha \colon M_{s(\alpha)} \to M_{t(\alpha)}$, for each arrow $\alpha \in Q_1$. If each vector space M_i is finite dimensional, we say that M is *finite dimensional*, and the *dimension vector* $\underline{\dim}\, M$ of M is the vector $(\dim M_i)_{i \in Q_0}$ of the dimensions of the vector spaces. For example, the representation in Figure 3 is a representation with dimension vector $(0, 1, 1, 1, 1, 0, 0)$ of the type $\mathbb{A}_7$ quiver in Figure 1.

Here $\overline{[4]} = \{3\}$, $\underline{[4]} = \{1, 2, 4\}$ $Q =$ 

Def (N. Reading)
Define the Q-Cambrian lattice on the set of triangulations of polygon Q.
$T_1 \lessdot T_2$ (T_i is covered by T_2)
if T_1 and T_2 differ by a diagonal flip $d_1 \to d_2$ where slope(d_1) < slope(d_2)

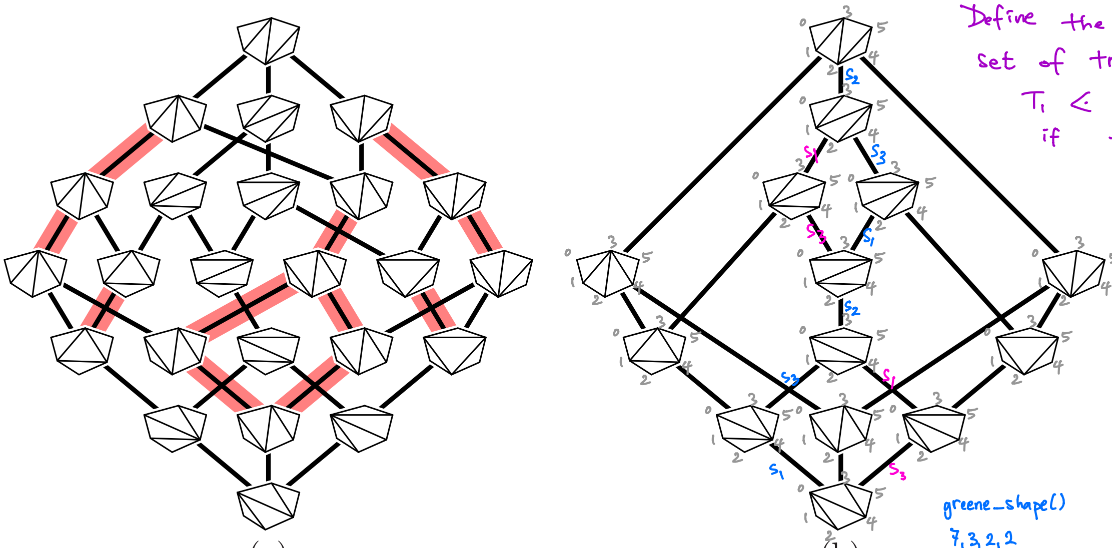


FIGURE 4. a: A non-Tamari permutations-to-triangulations map applied to every permutation in S_4. b: A non-Tamari Cambrian lattice

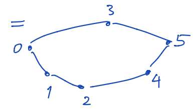
family of maps in the context of signed permutations. These families of maps arise quite naturally in the context of (equivariant) iterated fiber polytopes, as explained in [6] and [45, Section 4.3] and as summarized in [36, Sections 4, 6].

To generalize η, we alter the construction of the polygon Q by removing the requirement that the vertices 1 through $n+1$ be located below the horizontal line containing 0 and $n+2$. We keep the requirement that, for all i from 0 to $n+1$, the vertex i is strictly further left than the vertex $i+1$. Again we start with a path along the bottom edges of Q, and read the one-line notation of a permutation from left to right. When we read an entry whose corresponding vertex is on the bottom of Q, we **remove** that vertex from the path, as before. When we read an entry whose corresponding vertex is on the top of Q, we **insert** that vertex into the path. Figure 4.a shows this new permutations-to-triangulations map applied to all of the permutations in S_4, in the case where the vertices 1, 2, and 4 are on the bottom of Q and 3 is on the top. To avoid a profusion of notation, we use the symbol η to refer to any of the permutations-to-triangulations maps, tacitly assuming a choice of Q. We use the phrase "the Tamari case" to distinguish the original definition of Q and of η.

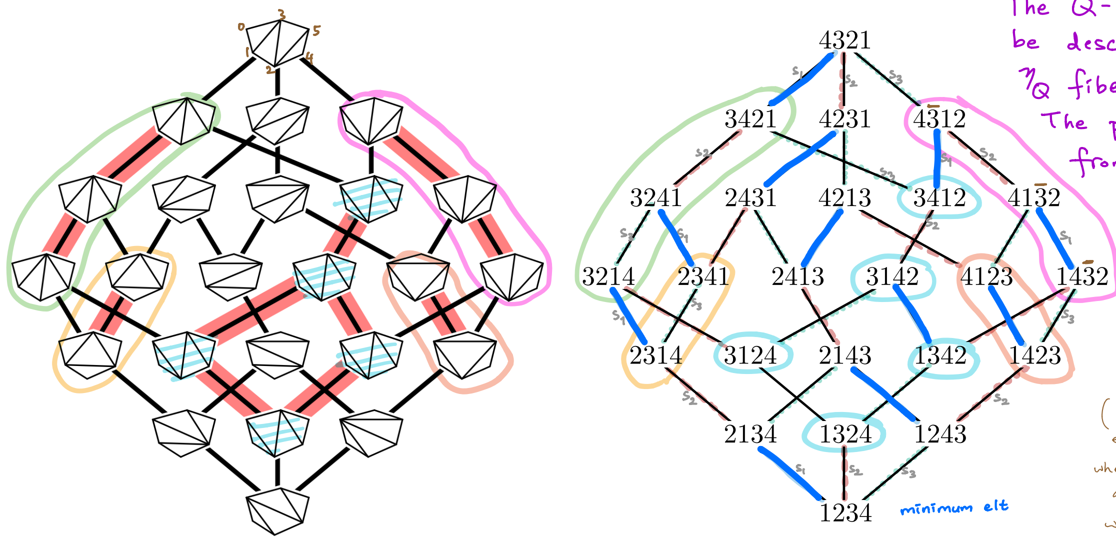
As another example, consider the case where all of the vertices 1 through n are **above** the line containing 0 and $n+2$. In this case, the symmetries of the problem imply that η has the same pattern-avoidance properties as described above, except that "312" is replaced by "231" throughout the description, and "132" is replaced by "213" throughout.

When some vertices are on top of Q and others are on bottom, as in the example of Figure 4.a, the behavior of the map is a mixture of the "231-behavior" and the "312-behavior," as we now explain. The locations, top or bottom, of the vertices are recorded by upper- or lower-barring the symbols from 1 to $n+1$. Thus, for example, we write $\overline{3}$ to indicate that the vertex 3 is on top of the polygon Q or we write $\underline{3}$ to indicate that 3 is on the bottom of Q. In [36, Proposition 5.7], it is shown that a permutation is a minimal element in its η-fiber if and only if it avoids

[36] Reading, Cambrian Lattices (06)

Here $\overline{[4]} = \{3\}$, $\underline{[4]} = \{1, 2, 4\}$ $Q =$ 

$s_1 := (12) = 2134$ $s_2 := (23) = 1324$ $s_3 := (34) = 1243$



(a) (b)

Thm (N. Reading)
The Q-Cambrian lattice can also be described as the set of η_Q fibers (preimages of η_Q). The poset relation is induced from the (right) weak order on the symmetric group

Note: The minimum elt in each fiber is $3\underline{1}\underline{2}$-avoiding and $\overline{2}31$-avoiding

(In this special case)
equivalently, 312-avoiding where the middle "2" is an even number, and 231-avoiding where the middle "2" is an odd number

See more details below

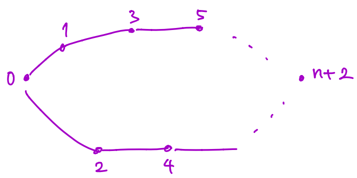
Thm (N. Reading) Let $\eta := \eta_Q$

- A permutation π is the minimum elt in its η-fiber iff π is $3\underline{1}\underline{2}$-avoiding and $\overline{2}31$-avoiding
- A permutation π is the maximum elt in its η-fiber iff π is $13\underline{2}$-avoiding and $\overline{2}13$-avoiding.

2.5 Bipartite Cambrian Lattices

Def ("Bipartite" Q-Cambrian lattice)

Consider the following Q. All odd numbers in $[n+1]$ are upper-barred, & all even numbers in $[n+1]$ are lower-barred.



(Note: where we put 1 and $n+1$ wouldn't make a difference in the pattern avoidance because 1 and $n+1$ are never the middle "2" in a pattern)

Then the Q-Cambrian lattice can be defined on the set

$$P = \left\{ \pi \in S_{n+1} \;\middle|\; \begin{array}{l} \pi \text{ avoids the pattern } 312 \text{ with an even "middle 2"} \\ \pi \text{ avoids the pattern } 231 \text{ with an odd "middle 2"} \end{array} \right\}$$

fyi: here we use the minimum elements of the η_Q fibers

where $x \leq y$ if

- $x w = y$ for some $w = s_{i_1} s_{i_2} s_{i_3} \cdots s_{i_\ell}$, where $i_k \in \{1, 2, \dots, n\}$
- $\operatorname{inv}(x) + \ell = \operatorname{inv}(y)$

In other words, there is a path from x to y in the (right) weak order which goes up at each step.

REU Exercise 9 (the bipartite analog of REU Exercise 8)

- Warm-up Consider a longest chain in the weak order

$$\mathrm{Id} \xrightarrow{s_1} \cdot \xrightarrow{s_3} \cdot \xrightarrow{s_2} \cdot \xrightarrow{s_1} \cdot \xrightarrow{s_3} \cdot \xrightarrow{s_2} w_0 = 4321$$

The $\binom{4}{2}+1 = 7$ permutations in this chain are

$$\begin{array}{lll}
s_1 s_3 s_2 s_1 s_3 s_2 & = & 4\ 3\ 2\ 1 \\
s_1 s_3 s_2 s_1 s_3 & = & 4\ 2\ 3\ 1 \\
s_1 s_3 s_2 s_1 & = & 4\ 2\ 1\ 3 \\
s_1 s_3 s_2 & = & 2\ 4\ 1\ 3 \\
s_1 s_3 & = & 2\ 1\ 4\ 3 \\
s_1 & = & 2\ 1\ 3\ 4 \\
\mathrm{Id} & = & 1\ 2\ 3\ 4
\end{array}$$

s_2, s_3, s_1, s_2, s_3, s_1

(i) Verify that each of the seven permutations in this chain is

$31\underline{2}$- avoiding and $\overline{2}31$-avoiding
"middle 2" is even "middle 2" is odd

For example, $s_1 s_3 s_2 = 2413$ has the subsequence 4,1,3 which fits the 312 pattern, but the "middle 2" is an odd number.

(ii) Verify that each of the seven permutations in this chain is

$13\underline{2}$- avoiding and $\overline{2}13$-avoiding.
"middle 2" is even "middle 2" is odd

- A longer warm-up

Verify that all $\binom{5}{2}+1 = 11$ permutations in the chain from $Id = \pi_{min} = 12345$ to $w_0 = \pi_{max} = 54321$

$$Id \xrightarrow{s_1} \cdot \xrightarrow{s_3} \cdot \xrightarrow{s_2} \cdot \xrightarrow{s_4} \cdot \xrightarrow{s_1} \cdot \xrightarrow{s_3} \cdot \xrightarrow{s_2} \cdot \xrightarrow{s_4} \cdot \xrightarrow{s_1} \cdot \xrightarrow{s_3} w_0 = 54321$$

(i) are $31\underline{2}$- avoiding and $\overline{2}31$-avoiding
"middle 2" is even "middle 2" is odd

(ii) are $13\underline{2}$- avoiding and $\overline{2}13$-avoiding.
"middle 2" is even "middle 2" is odd

$\binom{6}{2} = \frac{6 \cdot 5}{2} = 15$

- For S_6, have length $\binom{6}{2}$. Have $s_1, s_2, s_3, s_4, s_5 := (5,6)$

$$Id \xrightarrow{s_1} \cdot \xrightarrow{s_3} \cdot \xrightarrow{s_5} \cdot \xrightarrow{s_2} \cdot \xrightarrow{s_4}$$
$$\xrightarrow{s_1} \cdot \xrightarrow{s_3} \cdot \xrightarrow{s_5} \cdot \xrightarrow{s_2} \cdot \xrightarrow{s_4}$$
$$\xrightarrow{s_1} \cdot \xrightarrow{s_3} \cdot \xrightarrow{s_5} \cdot \xrightarrow{s_2} \cdot \xrightarrow{s_4} w_0 = 654321$$

- The REU Exercise 9 (i) (For S_{n+1}) Prove that all $\binom{n+1}{2}+1$ permutations in the chain

$$Id \xrightarrow{s_1} \cdot \xrightarrow{s_3} \cdot \xrightarrow{s_5} \cdot \cdots \cdot \xrightarrow{s_2} \cdot \xrightarrow{s_4} \cdot \xrightarrow{s_6} \cdot \cdots \Big|$$

the bars are for clarification only

$$\xrightarrow{s_1} \cdot \xrightarrow{s_3} \cdot \xrightarrow{s_5} \cdot \cdots \cdot \xrightarrow{s_2} \cdot \xrightarrow{s_4} \cdot \xrightarrow{s_6} \cdot \cdots \Big|$$

$\cdots$

$\xrightarrow{s_1} \xrightarrow{s_3} \cdots$ of length $\frac{n(n+1)}{2}$

are $31\underline{2}$- avoiding and $\overline{2}31$-avoiding.
"middle 2" is even "middle 2" is odd

(ii) Prove that these $\binom{n}{2}+1$ permutations

are $13\underline{2}$- avoiding and $\overline{2}13$-avoiding.
"middle 2" is even "middle 2" is odd

Note: May be more natural to use the chain

$$Id \rightarrow \cdots \xrightarrow{s_5} \xrightarrow{s_3} \xrightarrow{s_1} \xrightarrow{s_2} \xrightarrow{s_4} \xrightarrow{s_6} \cdots$$
$$\rightarrow \cdots \xrightarrow{s_5} \xrightarrow{s_3} \xrightarrow{s_1} \xrightarrow{s_2} \xrightarrow{s_4} \xrightarrow{s_6} \cdots$$

Computing λ, the Greene-Kleitman invariant for the bipartite Q Cambrian lattice

```
sage: A = CoxeterGroup(['A', n])   #Same as the symmetric group Sn+1
sage: c = [i for i in range(1,n+1,2)] + [i for i in range(2,n+1,2)]
      # for example, c = [1,3,5] + [2,4]  if n=5 for S6
sage: c = tuple(c)
sage: T = A.cambrian_lattice(c)
sage: T.greene_shape()                          tuple type
sage: Tam = A.cambrian_lattice((1, 2, ..., n))  # Tamari lattice for Sn+1
                                                  posets.TamariLattice(n+1)
```

Data and Conjectures

Denote the bipartite Q Cambrian lattice we describe above by

Camb_{alt} or Camb_{bi} or C_{alt} or ...

- For $n = 4, 5, 6, 7, 8$ $(S_5, S_6, S_7, S_8, S_9)$, Sage computes $\lambda_2 = \lambda_1 - 2$
  - Conjecture: this formula holds for all $n \geq 4$
- For $n = 6, 7, 8$ (S_7, S_8, S_9), Sage computes $\lambda_1 - \lambda_2 = 2$ and $\lambda_3 = \lambda_2$.
  - Conjecture: this formula holds for all $n \geq 6$
- For $n = 8$ (S_9), $\lambda_4 = \lambda_3 = \lambda_2$.
  - Conjecture: $\lambda_k = \lambda_{k-1} = \dots = \lambda_4 = \lambda_3 = \lambda_2$ for large enough n.

REU PROBLEM II (part 2) I think this is more promising than part 1

(i) Prove / disprove the conjecture for $\lambda_2 = \lambda_1 - 2$ for all $n \geq 4$.

Possible strategy: Imitate the Tamari lattice argument in Early's paper. the bipartite case is even easier

- Construct a union of two chains of size $\binom{n+1}{2} + \binom{n+1}{2}$

$$\Big[\underbrace{\binom{n+1}{2} + 1}_{\lambda_1} + \binom{n+1}{2} + 1\Big] - 2 = n(n-1)$$

typo fixed

- ~~To show that this union size is maximum, construct an antichain cover of Camb_{alt} consisting of antichains such that two of the antichains are singletons.~~ $\underbrace{\binom{n+1}{2}+1}_{\lambda_1}$
- This union size is maximum because the Cambrian lattice has the minimum elt (unique minimum elt) and the maximum elt.

- For the realization, use either the triangulations of polygon Q_{alt} or $31\underline{2}$-avoiding and $\overline{2}31$-avoiding permutations, which are the same as c-sortable elements (see later part of doc).

(ii) Prove / disprove the conjecture for $\lambda_3 = \lambda_2$ for all $n \geq 6$.

Possible strategy: Imitate the Tamari lattice argument in Early's paper.

~ ended Fri June 12, 2020 ~

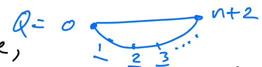
2.6 c-singletons

Def "eta"

Let $\eta_c := \eta_Q$.

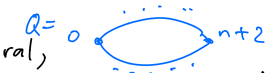
A permutation $\pi \in S_{n+1}$ s.t $\eta_c^{-1}(\eta(\pi)) = \{\pi\}$ is called a c-singleton.

Note: • If η_c is the Tamari type,



a permutation is a c-singleton iff it is (i) 312-avoiding and (ii) 132-avoiding.

• In general,



a permutation is a c-singleton iff it is (i) $31\underline{2}$-avoiding and $\overline{2}31$-avoiding and (ii) $13\underline{2}$-avoiding and $\overline{2}13$-avoiding.

[Lemma that you will need to use below]

If you follow a maximum-length chain from the minimum elt $123\cdots,n{+}1$ of the weak order to the maximum elt $w_0 = n{+}1, n, \ldots, 3, 2, 1$ such that every elt in the chain is a c-singleton,

$$s_{i_1} s_{i_2} \cdots s_{i_{\binom{n+1}{2}}}$$

then we can get a new maximum-length chain of c-singletons by performing the following commutation (or short braid move):

Replace $\quad s_{i_1} s_{i_2} \cdots s_k s_j \cdots s_{i_{\binom{n+1}{2}}}$ with

$\qquad\qquad s_{i_1} s_{i_2} \cdots s_j s_k \cdots s_{i_{\binom{n+1}{2}}}$

where $|j-k| \geq 2$.

Another Lemma

Every maximum-length chain of c-singletons can be achieved by applying a sequence of short braid moves from one maximum-length chain of c-singletons.

2.7 Maximum-length chains of c-singletons (bipartite c)

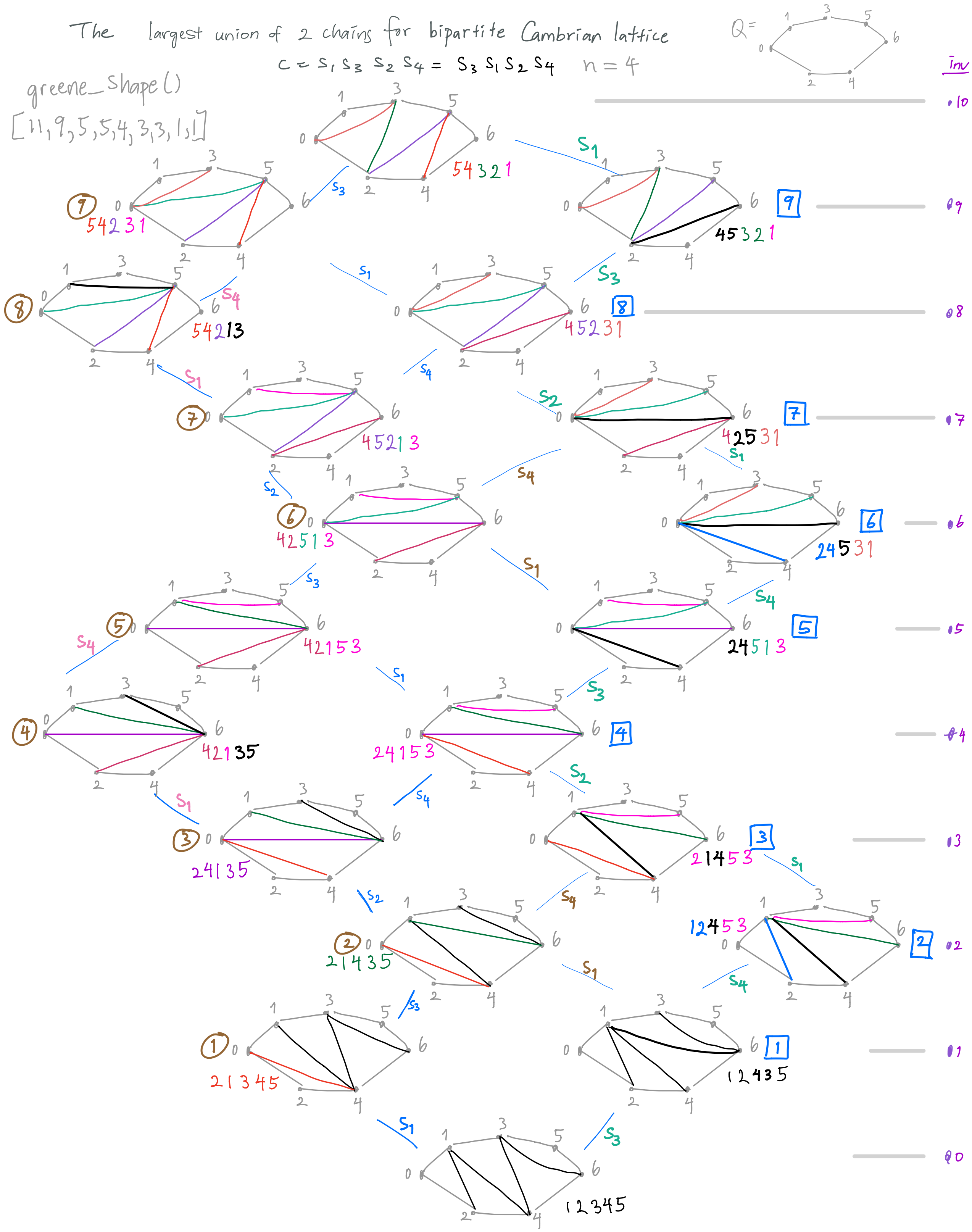
The largest union of 2 chains for bipartite Cambrian lattice

$c = s_1 s_3 s_2 s_4 = s_3 s_1 s_2 s_4$ $n = 4$

Q=

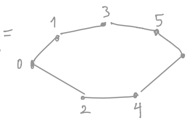
greene_Shape()

[11, 9, 5, 5, 4, 3, 3, 1, 1]



The largest union of 2 chains for bipartite Cambrian lattice

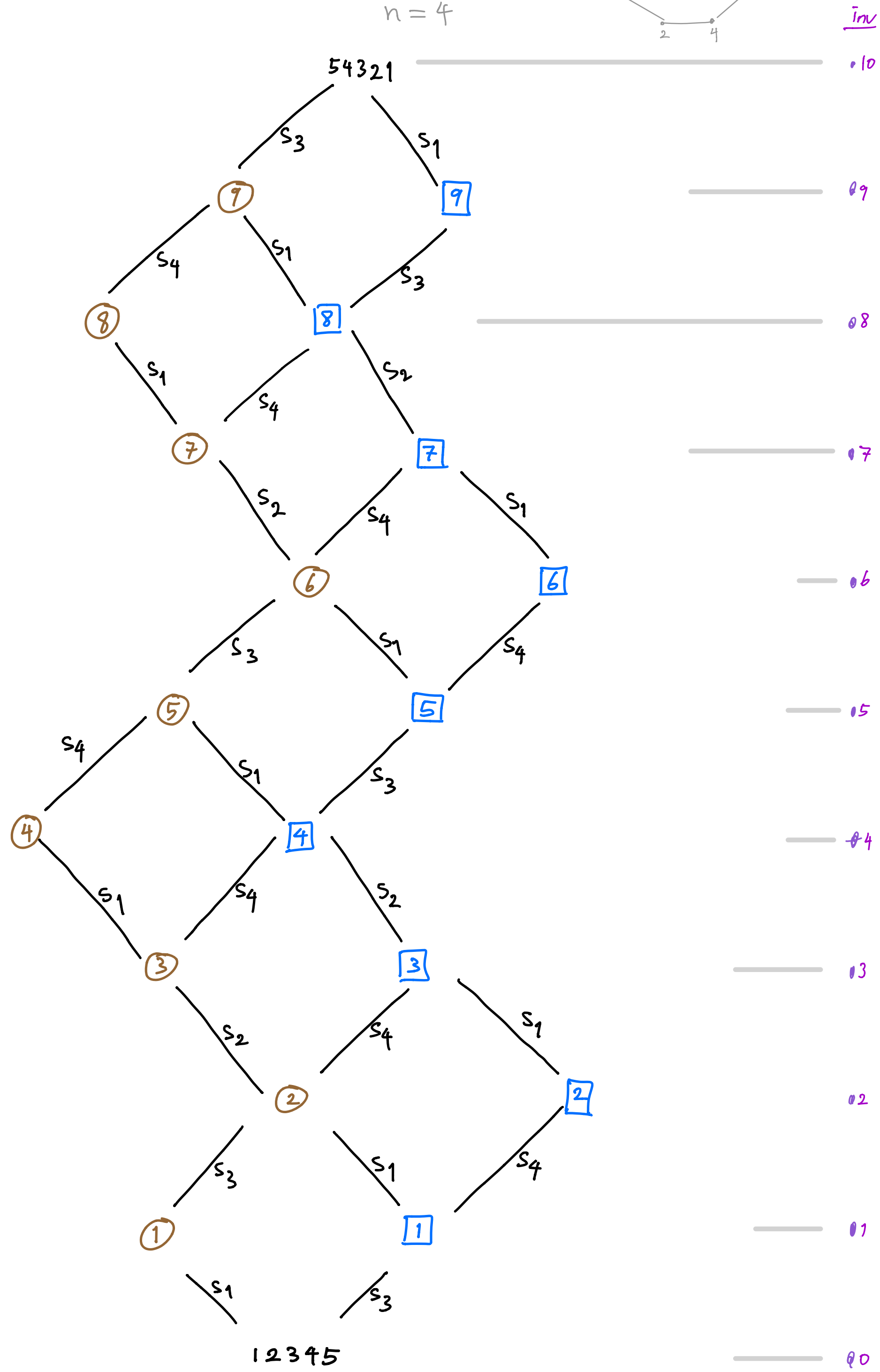
$n = 4$

Q= 

greene_Shape()

[11,9,5,5,4,3,3,1,1]

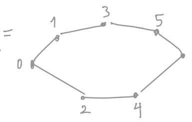
$c = s_1 s_3 s_2 s_4$
$= s_3 s_1 s_2 s_4$



The largest union of 2 chains for bipartite Cambrian lattice

$n=4$

$Q=$



greene_Shape()
[11,9,5,5,4,3,3,1,1]

$c = s_1 s_3 s_2 s_4$
$= s_3 s_1 s_2 s_4$

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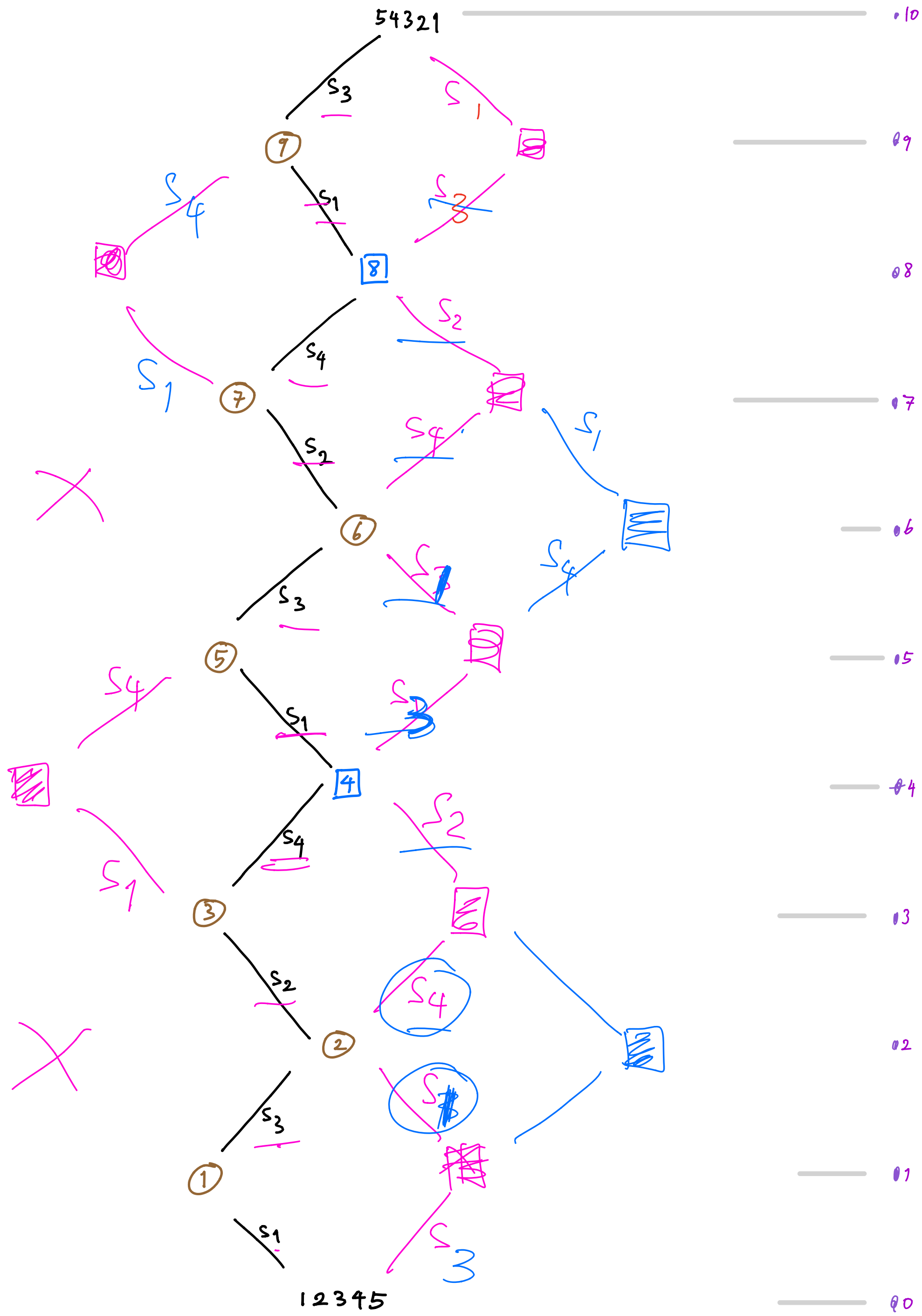
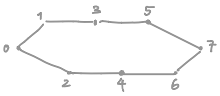


The chain from REU Exercise 9

for S_6 $n=5$

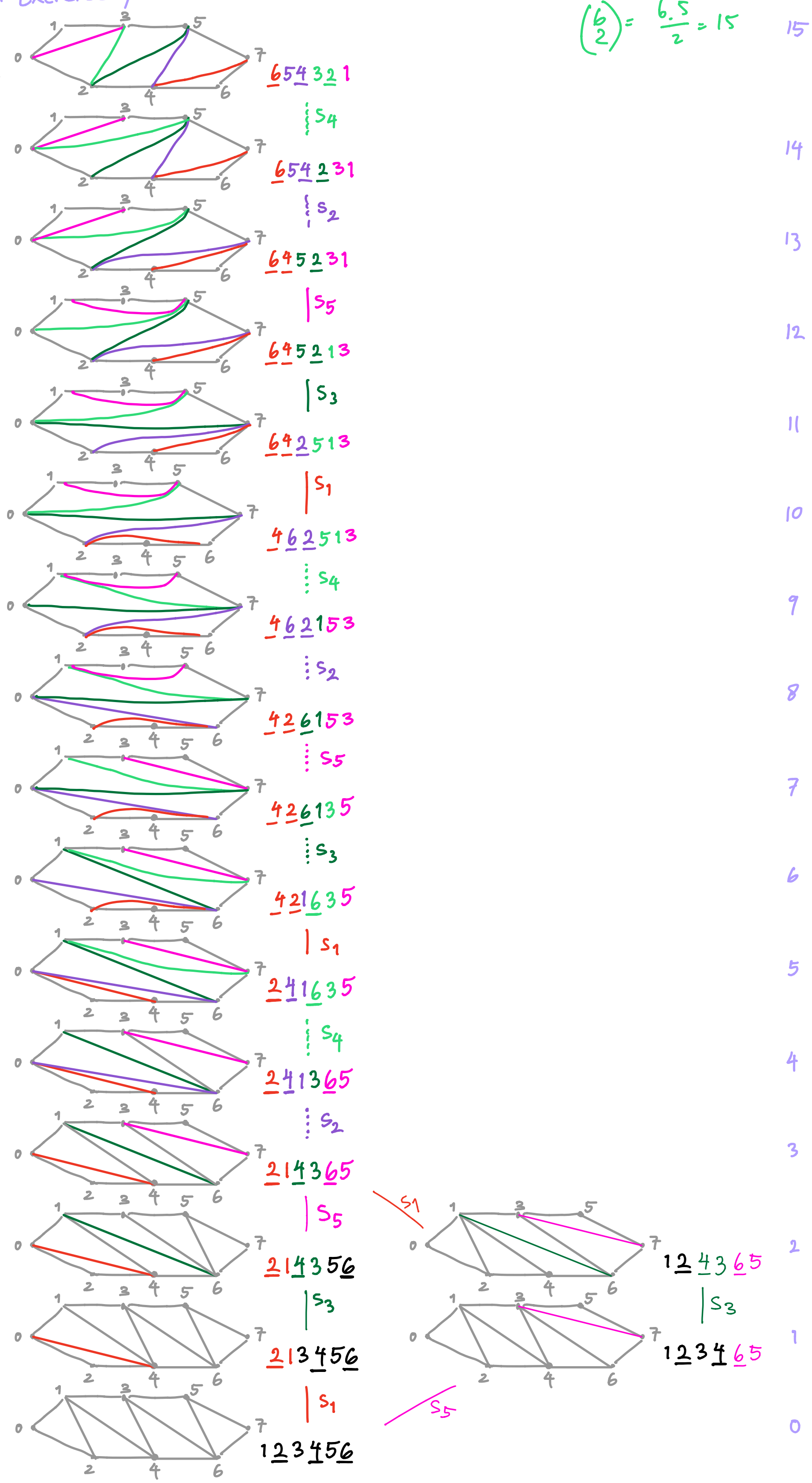


greene_shape()

[16, 14, 13, 9, 9, 8, 8, ...]

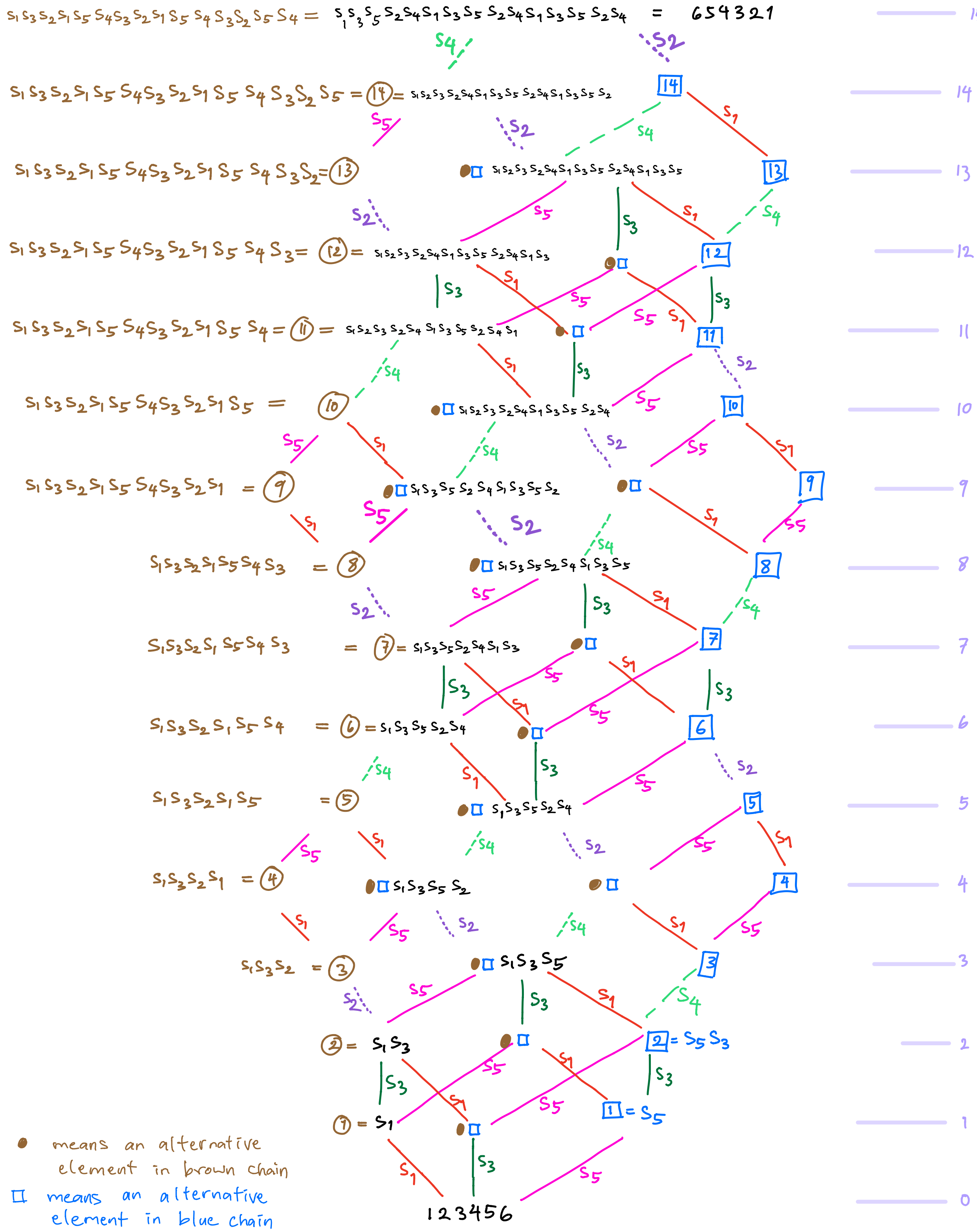
$c = s_1 s_3 s_5 s_2 s_4$
$= s_5 s_3 s_1 s_2 s_4$

$\binom{6}{2} = \frac{6.5}{2} = 15$



A (non-unique) largest union of 2 chains for bipartite A_5 Cambrian Lattice (S_6), of size 16+14

$C = s_1 s_3 s_5 s_2 s_4 = s_5 s_3 s_1 s_2 s_4$

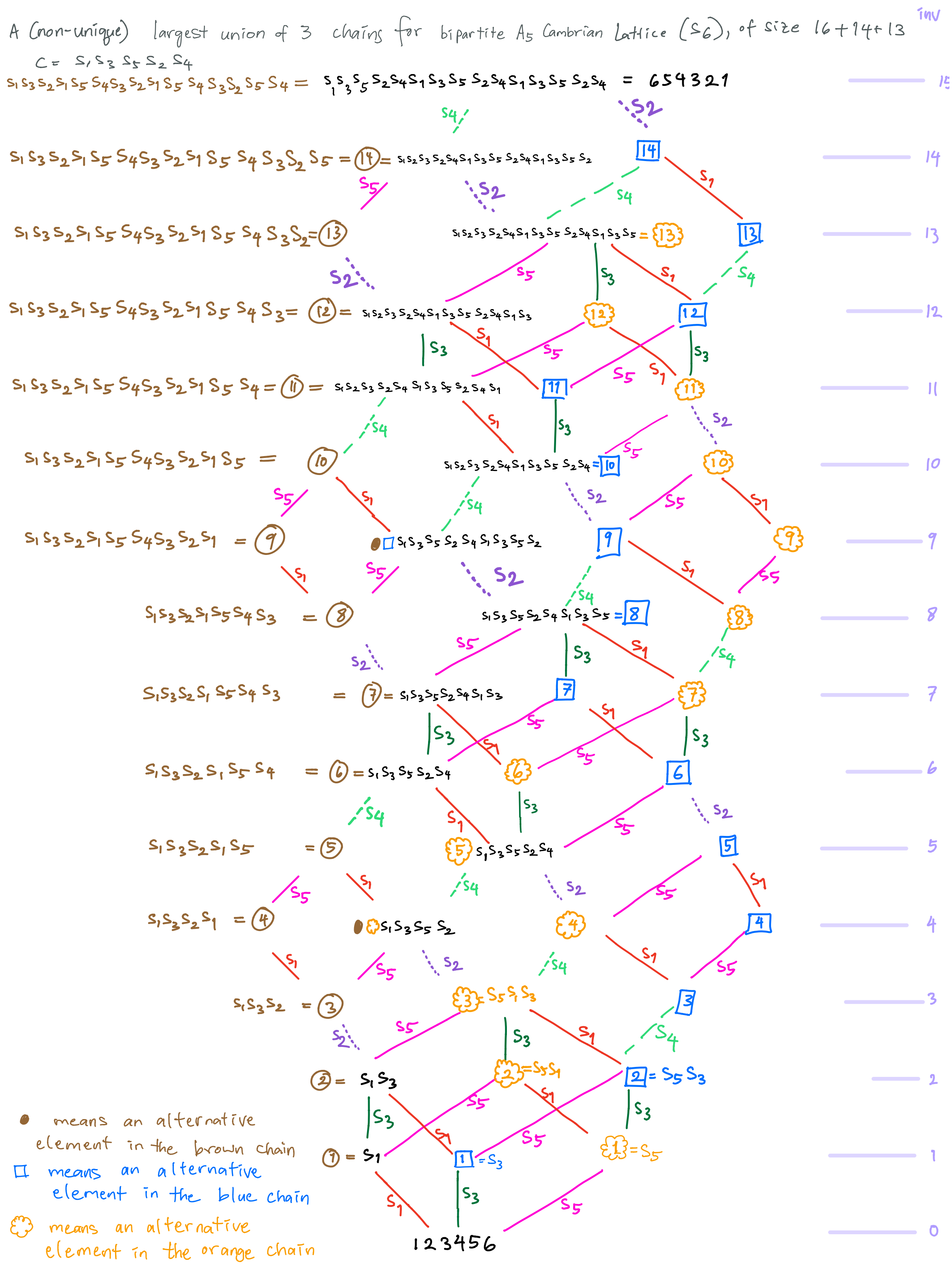


● means an alternative element in brown chain

◻ means an alternative element in blue chain

A (non-unique) largest union of 3 chains for bipartite A_5 Cambrian Lattice (S_6), of size 16+14+13

$c = s_1 s_3 s_5 s_2 s_4$



REU Exercise 10

The Greene-Kleitman invariant for Q_{alt} Cambrian lattice of type A_6 (S_7) is $[22, 20, 20, 18, 16, 15, \cdots]$.

(i) Starting from the length-21 chain

$$C = \xrightarrow{s_1}\xrightarrow{s_3}\xrightarrow{s_5}\xrightarrow{s_2}\xrightarrow{s_4}\xrightarrow{s_6}\xrightarrow{s_1}\xrightarrow{s_3}\xrightarrow{s_5}\xrightarrow{s_2}\xrightarrow{s_4}\xrightarrow{s_6}\xrightarrow{s_1}\xrightarrow{s_3}\xrightarrow{s_5}\xrightarrow{s_2}\xrightarrow{s_4}\xrightarrow{s_6}\xrightarrow{s_1}\xrightarrow{s_3}\xrightarrow{s_5}$$

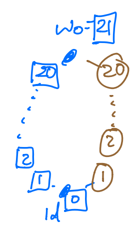
or

$$C = \xrightarrow{s_5}\xrightarrow{s_3}\xrightarrow{s_1}\xrightarrow{s_2}\xrightarrow{s_4}\xrightarrow{s_6}\xrightarrow{s_5}\xrightarrow{s_3}\xrightarrow{s_1}\xrightarrow{s_2}\xrightarrow{s_4}\xrightarrow{s_6}\xrightarrow{s_5}\xrightarrow{s_3}\xrightarrow{s_1}\xrightarrow{s_2}\xrightarrow{s_4}\xrightarrow{s_6}\xrightarrow{s_5}\xrightarrow{s_3}\xrightarrow{s_1},$$

sketch all maximum-length chains of c-singletons by applying commutation (short braid) moves.

(ii)

- The largest union (size 22+20) of 2 chains must be contained in the sketch, since the number 22+20 means that this union must be the union of two maximum-length chains of c-singletons.



- The largest union (size 22+20+20) of 3 chains must also be contained in the sketch, since the number 22+20+20 means that this union must be the union of three maximum-length chains of c-singletons.

- Based on the sketch,

  Conjecture two disjoint chains for general n which union has $(n+1)n$ elements.

  Conjecture three disjoint chains for general n which union has $(n+1)n + \binom{n+1}{2} - 2$.

~ the end ~