Last updated 5 pm Thurs, June 11, 2020

Lectures given
Mon, June 8, 2020
Thurs, June 11, 2020

Sec. 2.1 Description of Tamari lattice

FROM THE TAMARI LATTICE TO CAMBRIAN LATTICES AND BEYOND

Review def of poset, chains, antichains in notes of Problem I, part 2.

NATHAN READING

Abstract. In this chapter, we trace the path from the Tamari lattice, via lattice congruences of the weak order, to the definition of Cambrian lattices in the context of finite Coxeter groups, and onward to the construction of Cambrian fans. We then present sortable elements, the key combinatorial tool for studying Cambrian lattices and fans. The chapter concludes with a brief description of the applications of Cambrian lattices and sortable elements to Coxeter-Catalan combinatorics and to cluster algebras.

Def • An elt x of a poset P is minimal (resp. maximal) if there is no $z \in P$ for which $z \lneq x$ (resp. $x \gneq z$)

• $x \in P$ is the minimum elt if $x \leq z$ $\forall z \in P$

• $x \in P$ is the maximum elt if $x \geq z$ $\forall z \in P$

1. A map from permutations to triangulations

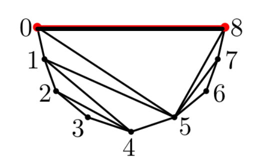
The road from the Tamari lattice to Cambrian lattices starts with a simple map from the set S_{n+1} of permutations of $\{1, \ldots, n+1\}$ to the set of triangulations of a convex polygon with $n+3$ vertices. This map connects the Tamari lattice to the weak order on permutations, and opens the door to understanding the Tamari lattice in a broader lattice-theoretic context.

One of the many realizations of the Tamari lattice is as a partial order on triangulations of a convex polygon. Specifically, take Q to be a convex $(n+3)$-gon in the plane and identify the vertices of Q with the numbers $0, 1, \ldots, n+1, n+2$. We require that the vertices 0 and $n+2$ be on a horizontal line, with 0 to the left and with all other vertices below that line. Furthermore, we require that the vertices 1 through $n+1$ be placed so that, for all i from 0 to $n+1$, the vertex i is strictly further left than the vertex $i+1$. A correct construction of Q, for $n = 7$, is shown in the top-left picture of Figure 2 for the case $n = 7$.

E.g. For S_{n+1}, let

$Q :=$

A *triangulation* of Q is a tiling of Q by triangles whose vertices are contained in the vertex set of Q. The triangulation is specified by the collection of n diagonals of Q appearing as edges of the triangles. A *diagonal flip* on a triangulation of Q is the operation of removing a diagonal of the triangulation to create a quadrilateral from two triangles, and then inserting the other diagonal of the quadrilateral to create a new triangulation. The Tamari lattice is a partial order on triangulations of Q whose cover relations are given by diagonal flips. The two triangulations in the cover differ by exactly one diagonal of Q, and the higher triangulation in the cover relation is the one in which this diagonal has larger slope. The Tamari lattice, for $n = 3$, is shown in Figure 1.



A triangulation
$\{(0,5),(1,4),(1,5),(2,4),(5,7),(5,8)\} \cup \{(0,1),(2,3),\ldots,(7,8),(0,8)\}$

the labels of the edges are higher

This definition of the Tamari lattice highlights its connection to the associahedron. Since the vertices of the associahedron can be labeled by triangulations of a fixed convex polygon such that edges are given by diagonal flips, the Hasse diagram of the Tamari lattice is isomorphic to the 1-skeleton of the associahedron.

4 is the # of largest antichain

2010 *Mathematics Subject Classification.* 20F55, 06B10, 05E15.

$\lambda(\text{Tamari } 4) = (7,3,3,1)$

Denote $S_j := (j, j+1)$ for all $j = 1, \ldots, n-1$.

$S_1 := (12) = 2134$ $S_2 := (23) = 1324$ $S_3 := (34) = 1243$

Permutations below are in one-line notation

In Sage Tamari lattice 4

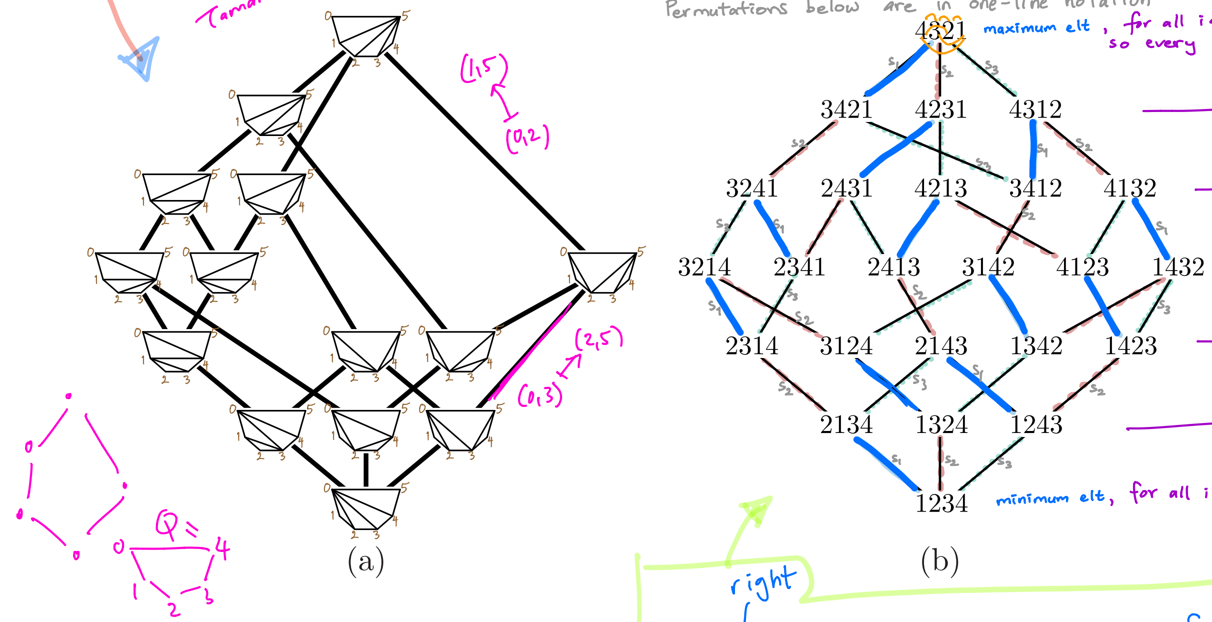


FIGURE 1. a: The Tamari lattice. b: The (right) weak order on permutations in S_n.

path along all bottom edges

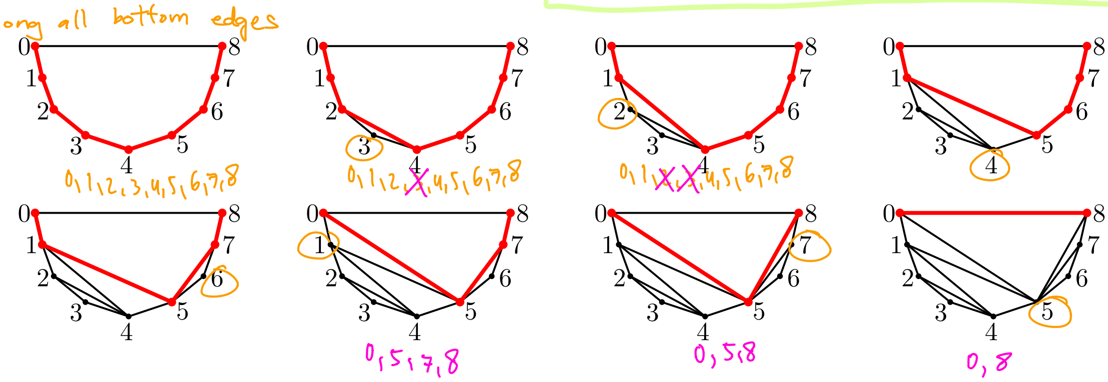


FIGURE 2. The triangulation $\eta(3246175)$

To define the (right) weak order, we first write permutations in one-line notation, meaning that we represent a permutation x of $\{1, \ldots, n+1\}$ by the sequence $x_1 x_2 \cdots x_{n+1}$, where x_i means $x(i)$. There is a cover relation $x \lessdot y$ in the weak order whenever the one-line notations of x and y differ only by swapping a pair of adjacent entries. The permutation x is the one in which the two entries appear in numerical order, and y is the permutation in which the two entries appear out of order. For example, the weak order on S_4 is shown in Figure 1.b.

Note: If $\pi \in S_n$, $\pi(i, i+1)$ is the permutation π with entries π_i and π_{i+1} swapped.

We now define a map η (eta) from S_{n+1} to the set of triangulations of Q. Start with a path along the bottom edges of Q, as shown in the first frame of Figure 2. Given a permutation $x \in S_{n+1}$, read from left to right in the one-line notation for x. For each entry, create a new path by deleting the corresponding vertex from the old path. The triangulation $\eta(x)$ is defined by the union of the sequence of paths, as illustrated in Figure 2 for the permutation with one-line notation 3246175. Figure 3.a shows the result of applying η to every permutation in S_4. The shaded edges indicate covering pairs in the weak order which map to the same triangulation.

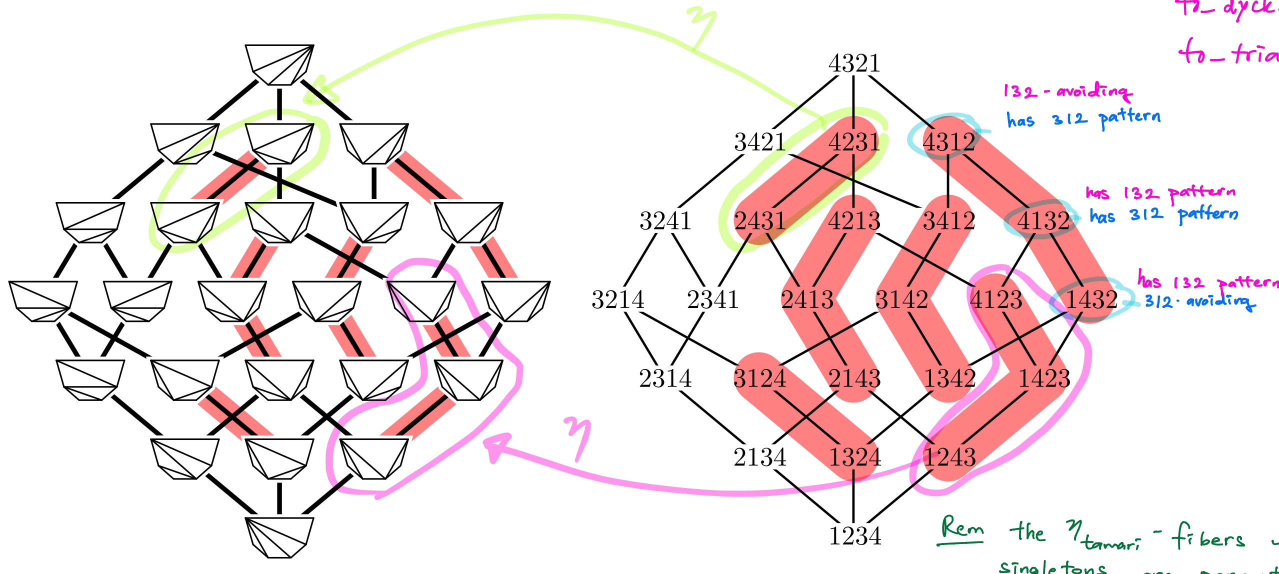
η is a surjection

This map and similar maps have appeared in many papers, including [6, 7, 29, 30, 36, 50]. The map can be seen in a broader context in the chapter by Rambau and Reiner [31] in this volume, specifically by giving some thought to [31, Theorem 9] and the accompanying figure.

[6] Billera - Sturmfels "Iterated Fiber polytopes" (94)
[7] Björner - Wachs "Shellable nonpure complexes & posets II" (97)
[29-30] Loday - Ronco "... Planar binary trees" (98-02)

arxiv.org

Sage
Permutation ↦ to_dyck_path ↦ to_triangulation



(a) (b)

Rem the η_{tamari}-fibers which are singletons are permutations which are both 312 and 132 avoiding.

FIGURE 3. a: The map η applied to every permutation in S_4. b: The Tamari congruence on S_4.

Go to next page

Björner and Wachs [7, Section 9] studied a map τ from permutations to binary trees that is, up to a standard bijection from triangulations to binary trees, identical to η. We describe their results in terms of the map η. First, the fiber $\eta^{-1}(\Delta)$ of each triangulation Δ is a non-empty interval in the weak order on S_{n+1}. A permutation is the minimal element in its η-fiber if and only if it avoids the pattern 312. That is, a permutation x is minimal in its fiber if and only if there is no sequence of three (not-necessarily adjacent) entries in the one-line notation for x such that the largest of the three is first, followed by the smallest of the three, and finally the median-valued. Similarly, a permutation is the maximal element in its η-fiber if and only if it avoids the pattern 132. For example, comparing Figures 1.b and 3.a, we see that the permutation 4213 is not the minimal element of its η-fiber, and indeed, the sequence 413 (or the sequence 423) is an instance of the pattern 312 in the permutation 4213. However, 4213 is the maximal element in its η-fiber because it avoids the pattern 132.

Björner and Wachs also showed that the weak order and the Tamari lattice are closely related. Specifically, the restriction of the weak order to 312-avoiding permutations is a sublattice of the weak order, and the restriction of η to this sublattice is an isomorphism from the sublattice to the Tamari lattice. This is readily seen in the case of S_4 by inspection of Figures 1 and 3.a. The sublattice of the weak order consisting of 312-avoiding permutations (and thus the Tamari lattice) is also a quotient of the weak order in an order-theoretic sense. Indeed, the results of [7] go most of the way to establishing something stronger: As we will see in Section 2, the Tamari lattice is a lattice quotient (i.e. a lattice-homomorphic image) of the weak order, because the map η is a lattice homomorphism. This is the key insight that leads to the notion of Cambrian lattices.

Before we shift the discussion to lattice theory, we give a generalization, in a more combinatorial direction, of the map η. We will see in Section 3 that this generalization is also an essential step towards Cambrian lattices. The generalization, which was exploited in [36], draws on the description in [45, Section 4.3] of a similar

[36] Reading, Cambrian Lattices (06)   [45] Reiner, Equivariant fiber polytopes (02)

Def If $x \le y$, the set $[x,y]$ of all elts z of P satisfying $x \le z \le y$ is called the (closed) interval between x and y.
Here x is the minimum elt of $[x,y]$ and y is the maximum elt of $[x,y]$

Thm (for Tamari lattice) Let $\eta := \eta_{tamari}$.

- The fiber (preimage of Δ) $\eta^{-1}(\Delta)$ of each triangulation Δ is a non-empty interval in the weak order on S_{n+1}.
- A permutation π is the minimum elt in its η-fiber iff π is 312-avoiding.
- A permutation π is the maximum elt in its η-fiber iff π is 132-avoiding.

Rem We just saw three realizations of the Tamari lattice (triangulations, 312-avoiding perms, 132-avoiding perms).
There are many (maybe 100) known realizations of the Tamari lattice.
The papers by Early and Lim & Zhang treat the elements as integer tuples (under Literature page "Tamari (lattices")

Def The length of a chain C in a poset is the number of elements in C minus 1.

REU Exercise 8 (Everyone should make sure they can do Exercise 8)

- Warm-up Consider the chain from $\pi_{min} = 1234$ to $w_0 := \pi_{max} = 4321$

$$Id = \pi_{min} \xrightarrow{s_1} \cdot \xrightarrow{s_2} \cdot \xrightarrow{s_3} \cdot \xrightarrow{s_1} \cdot \xrightarrow{s_2} \cdot \xrightarrow{s_1} 4321 \quad \text{in } S_4.$$

This chain is of length $\binom{4}{2} = 6$ and has $\binom{4}{2}+1 = 7$ permutations

$s_1 s_2 s_3 s_1 s_2 s_1$	=	4321
$s_1 s_2 s_3 s_1 s_2$	=	3421
$s_1 s_2 s_3 s_1$	=	3241
$s_1 s_2 s_3$	=	2341
$s_1 s_2$	=	2314
s_1	=	2134
Id	=	1234

swap position 2 and 3
swap position 1 and 2
swap position 3 and 4

(i) Verify that these $\binom{4}{2}+1$ permutations are 312-avoiding.
(ii) Verify that these $\binom{4}{2}+1$ permutations are 132-avoiding.

- A longer warm-up

Verify that all $\binom{5}{2}+1 = 11$ permutations in the chain from $Id = \pi_{min} = 12345$ to $w_0 = \pi_{max} = 54321$

$$Id \xrightarrow{s_1} \cdot \xrightarrow{s_2} \cdot \xrightarrow{s_3} \cdot \xrightarrow{s_4} \cdot \xrightarrow{s_1} \cdot \xrightarrow{s_2} \cdot \xrightarrow{s_3} \cdot \xrightarrow{s_1} \cdot \xrightarrow{s_2} \cdot \xrightarrow{s_1} w_0 = 54321$$

are also (i) 312-avoiding and (ii) 132-avoiding.

- The REU Exercise: (i) Prove that all $\binom{n+1}{2}+1$ permutations of S_{n+1} in the chain

$$Id \xrightarrow{s_1} \xrightarrow{s_2} \xrightarrow{s_3} \cdots \xrightarrow{s_n}$$
$$\xrightarrow{s_1} \xrightarrow{s_2} \xrightarrow{s_3} \cdots \xrightarrow{s_{n-1}}$$
$$\xrightarrow{s_1} \cdots \xrightarrow{s_{n-2}}$$
$$\vdots$$
$$\xrightarrow{s_1} \xrightarrow{s_2}$$
$$\xrightarrow{s_1}$$

are 312-avoiding.

(ii) Prove that they are also 132-avoiding.

— end Monday, June 8, 2020 —

2.2 Greene-Kleitman invariants of the Tamari lattices

REU Exercise 8 gives a proof to the following.

Proposition:
The length of a longest chain in the Tamari lattice Tamari_n is the same as the length of a longest chain in S_n, $\binom{n}{2} = \frac{n(n-1)}{2}$.
The size of a longest chain in the Tamari lattice Tamari_n is $\binom{n}{2}+1$.

Recall Greene-Kleitman invariant / partition

Thm (Greene, Kleitman, "The structure of Sperner k-families" 1976)
- Let $A_k :=$ the size of a largest union of k chains of P
- Let $D_k :=$ the size of a largest union of k antichains of P.
- Let $\lambda_k = A_k - A_{k-1}$ for all k, and $\lambda := (\lambda_1, \lambda_2, \dots)$
- Let $\mu_k = D_k - D_{k-1}$ for all k, and $\mu := (\mu_1, \mu_2, \dots)$

Then λ and μ are weakly decreasing, and they are conjugate.

Here λ is often called the Greene-Kleitman invariant/partition of P)

```
sage: T5 = posets.TamariLattice(5)
sage: T5.greene_shape()
[11, 7, 6, 5, 5, 3, 2, 2, 1]
```

← corresponds to S_5

← gives $\lambda(\mathrm{Tamari}\ 5)$

Def A chain cover of P is a collection of disjoint chains so that their union is P
An antichain cover of P is a collection of disjoint antichains so that their union is P
The size of a chain cover is the number of chains in it.

Dilworth's Theorem

The width of P (the size of a largest antichain) is the smallest size of a chain cover.
i.e. $D_1 = \#$ of positive parts of λ

Dual of Dilworth's Theorem (easier to prove directly)

The size of a longest chain of P is the smallest size of an antichain cover.
i.e. $A_1 = \#$ of positive parts of μ.

↳ Optional task: Watch videos of proofs of these two facts.
Link under "Dilworth's Theorem" on "Literature page"

Rem

By Proposition above, the size of a tallest chain in Tamari_n is $\binom{n}{2}+1$,

so the first row of $\lambda(\text{Tamari}_n)$ is $\lambda_1(\text{Tamari}_n) = \binom{n}{2}+1$.

Thm [Thm 1.1–1.2, Early 2003] Note: typo in Thm 1.1–1.2, written prior to SageMath

For $n \geq 4$, $\lambda_1(\text{Tamari}_n) - \lambda_2(\text{Tamari}_n) = 4$

For $n \geq 6$, $\lambda_2(\text{Tamari}_n) - \lambda_3(\text{Tamari}_n) = 2$

Thm [Thm 3.7, Lim & Zhang 2008 unpublished student project]

For $n \geq 8$ (or a large enough n),

there are formulas for $\lambda_4(\text{Tamari}_n)$ and $\lambda_5(\text{Tamari}_n)$

$\boxed{\text{REU PROBLEM II, part 1}}$ (I think part 2 of PROBLEM II is more promising)

(i) Compute $\lambda_k(\text{Tamari}_n)$ for $k=6$, or more.

First try the same technique as Early's and Lim & Zhang's.

(ii) Compute the $\underbrace{\text{width}}_{\text{size of largest antichain}}$ of Tamari_n. The first few numbers 2, 4, 9, 22, 61 are not in OEIS.org

- One way to compute the size of a largest antichain is to compute the number of positive parts of λ, but maybe you could come up with other methods.

Possible steps toward PROBLEM II (part 1):

1. Study Early's 2003 8-pg paper "Chain lengths in the Tamari lattice"
2. Rephrase Early's argument using triangulations or 312-avoiding permutations or 132-avoiding permutations. Does it make the argument nicer to read?
3. Study and study the accuracy of Lim & Zhang's report "Previously unknown parts of the Greene–Kleitman partition for the Tamari lattice"

E.g. Thm 3.7 is wrong for $n=8$, but this could be a minor typo.

```
sage: T5 = posets.TamariLattice(5)
sage: T5.greene_shape()
[11, 7, 6, 5, 5, 3, 2, 2, 1]
```

← gives $\lambda(\text{Tamari } 5)$

Possible alternative project for PROBLEM II (part 1)

Write a proof using the same methods as Early's and Lim-Zhang's but using a different realization of the Tamari lattice (possibilities: triangulations, 312-avoiding permutations, Young diagrams which fit in a rectangular shape)

2.3 More properties of the map η_{tamari}

Ref Sec 5 of Reading's "Cambrian Lattices" 2005

Def $\pi \in S_n$ has pattern 312 if there exist $1 \le i < j < k \le n$ with $\pi_j < \pi_k < \pi_i$

For example, both $5\,3\,1\,6\,2\,7\,4 \in S_7$ and $3\,1\,6\,4\,7\,2\,5 \in S_7$ have pattern 312

$\pi \in S_n$ has pattern 132 if there exist $1 \le i < j < k \le n$ with $\pi_i < \pi_k < \pi_j$

For example, both $5\,3\,1\,2\,6\,7\,4 \in S_7$ and $3\,1\,4\,6\,7\,2\,5 \in S_7$ have pattern 132

Def If $\pi \in S_n$ has pattern 312 witnessed by $\pi_j < \pi_k < \pi_i$ for some $1 \le i < j < k \le n$.
If $j = i+1$, then altering π by switching the entries π_i and π_{i+1} is called a $312 \longrightarrow 132$ move

[Special case of Prop 5.3, Reading's "Cambrian Lattices" 2005]

Suppose $x \lessdot y$ in the weak order. Let $\eta := \eta_{tamari}$
x is covered by y

Then $\eta(x) = \eta(y)$ iff

x is obtained from y by a $312 \longrightarrow 132$ move.

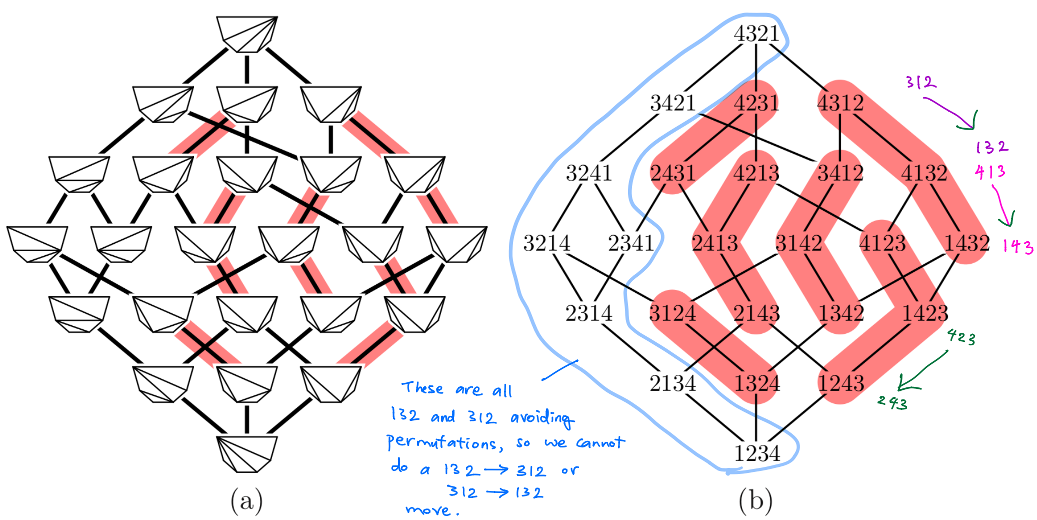
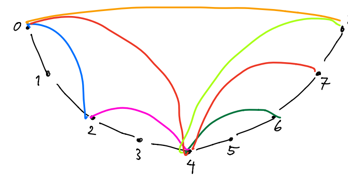


FIGURE 3. a: The map η applied to every permutation in S_4. b: The Tamari congruence on S_4.

Examples of $3\,1\,2 \to 1\,3\,2$ move from y to x

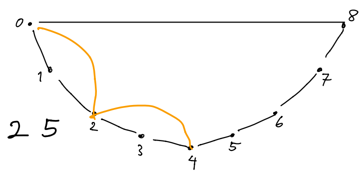
(1) $y = 5\ 3\ 1\ 6\ 2\ 7\ 4$

$x = 5\ 3\ 1\ 2\ 6\ 7\ 4$



(2) $y = 3\ 1\ 6\ 4\ 7\ 2\ 5$

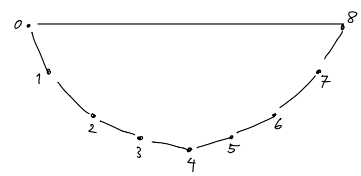
$x = 3\ 1\ 4\ 6\ 7\ 2\ 5$



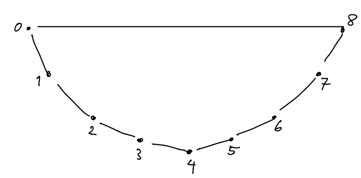
$\boxed{1\,3}\ 6\ 4\ 7\ 2\ 5, \quad 3\ 1\ 6\ 4\ \boxed{2\,7}\ 5$

Examples of covering $y \gtrdot x$ where x is not obtained from y by a $3\,1\,2 \to 1\,3\,2$ move

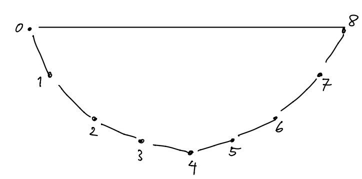
(3) $y = 4\ 5\ 3\ 1\ 6\ 2\ 7$



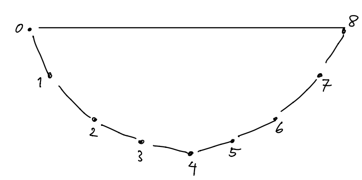
$x = 4\ 5\ 3\ 1\ 2\ 6\ 7$



(4) $y = 3\ 5\ 1\ 6\ 4\ 7\ 2$



$x = 3\ 5\ 1\ 4\ 6\ 7\ 2$



[Special case of Lemma 5.5, "Cambrian Lattices"] Let $\eta := \eta_{tamari}$.

If π contains a pattern 312,

then π contains an instance of the pattern 312

such that the "3" and the "1" are adjacent in π.

So, if π contains 312, we can always perform a $312 \to 132$ move to go down one level (inversion number) in the weak order and still stay in the same fiber $\eta^{-1}(\eta(\pi))$.

Recall we know this is an interval in the weak order

We can keep applying a $312 \to 132$ move to permutations which have the pattern 312, until we get to the minimum element of the interval $\eta^{-1}(\eta(\pi))$. Recall that a permutation is 312-avoiding iff it's the minimum element of a fiber of η.

— the end —