Notes for Bruce Sagan Oct 28, 2020 DAC BIRS

Def (How to associate a Coxeter element to an orientation of a type A Dynkin quiver)

Ref: "c-sortable elements" papers

Let $c = s_{i_1} s_{i_2} \cdots s_{i_n}$ be a Coxeter element in the Symmetric group S_{n+1}.

Then we can write c in cycle notation as

Let these integers be "upper-barred" integers

$$c = (1, \underbrace{d_1, d_2, \ldots, d_\ell}, n+1, \overbrace{u_k, u_{k-1}, \ldots, u_1}).$$

Let these integers be "lower-barred" integers

Define $Q(c)$ to be an orientation of $1 - 2 - \cdots - n$

as follows: orient $i-1 \leftarrow \underline{i}$ if i is a lower-barred integers

orient $i-1 \rightarrow \bar{i}$ if i is an upper-barred integer.

Example 1 $c = s_1 s_2 s_3 = (12)(23)(34) = (1\underline{2}\underline{3}4) = \begin{pmatrix} 1 & 2 & 3 & 4 \\ 2 & 3 & 4 & 1 \end{pmatrix}$

Read right to left like function composition

Only lower-barred integers, no upper-barred integers

Then $Q(s_1 s_2 s_3) = 1 \leftarrow \underline{2} \leftarrow \underline{3}$

Interval $[Id, s_1 s_2 s_3]$ in the right weak order is a chain:

$s_1 s_2 s_3 = 2341$
|
$s_1 s_2 = 2314$
|
$s_1 = 2134$
|
$Id = 1234$

F-polynomial:

$1 + y_1 + y_1 y_2 + y_1 y_2 y_3$

$\bar{3}$ is upper barred

Example 2 $c = s_1 s_3 s_2 = (12)(34)(23) = (1\underline{2}4\bar{3}) = \begin{matrix} 1 & 2 & 3 & 4 \\ 2 & 4 & 1 & 3 \end{matrix}$

$\underline{2}$ is lower barred

Then $Q(s_1 s_3 s_2) = 1 \leftarrow \underline{2} \rightarrow \bar{3}$

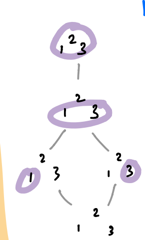
Interval $[Id, s_1 s_3 s_2]$ in the right weak order

$s_3 s_1 s_2 = s_1 s_3 s_2 = 2413$

$s_1 s_3 = 2143$

$2134 = s_1$ $\quad$ $s_3 = 1243$

Id



F-polynomial:

$y_1 y_2 y_3 +$

$y_1 y_3 +$

$y_1 + y_3$

$+$

1

[Çanakçı & Schroll 2018/2020 "Lattice bijections for string modules, snake graphs and the weak Bruhat order"] says:

There is a lattice bijection between

the interval $[\mathrm{Id}, c]$ in the weak order

└ a Coxeter element

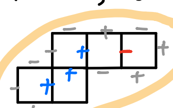
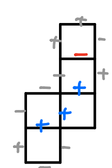
and

the lattice of order ideals / order filters of the (fence) poset whose Hasse diagram is $Q(c)$.

Summary of relevant parts of

[Çanakçı & Schroll 2018/2020 "Lattice bijections for string modules, snake graphs and the weak Bruhat order"]

Let G be a snake graph, that is, a sequence of $+$ and $-$

e.g. $+++-$ corresponds to  or  (depending on your convention)

use this in this note

Associate a quiver $Q(G)$ to G in the obvious way:

$1 \to 2 \to 3 \to 4 \leftarrow 5$ or $1 \leftarrow 2 \leftarrow 3 \leftarrow 4 \to 5$

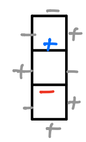
In this note, let's pick this convention

Now build the lattice $\mathcal{L}$ of order ideals / order filters of $Q(G)$ or (equivalently) the lattice of perfect matchings of G.

[ГS] Lemma 3.16 says: Treating each edge of $\mathcal{L}$ as a simple reflection, let $c = s_{i_1} \cdots s_{i_n}$ and $c' = s_{i_1} \cdots s_{i_n}$ be any two maximal chains of $\mathcal{L}$.
Then $c = c'$ as an element in S_{n+1}.

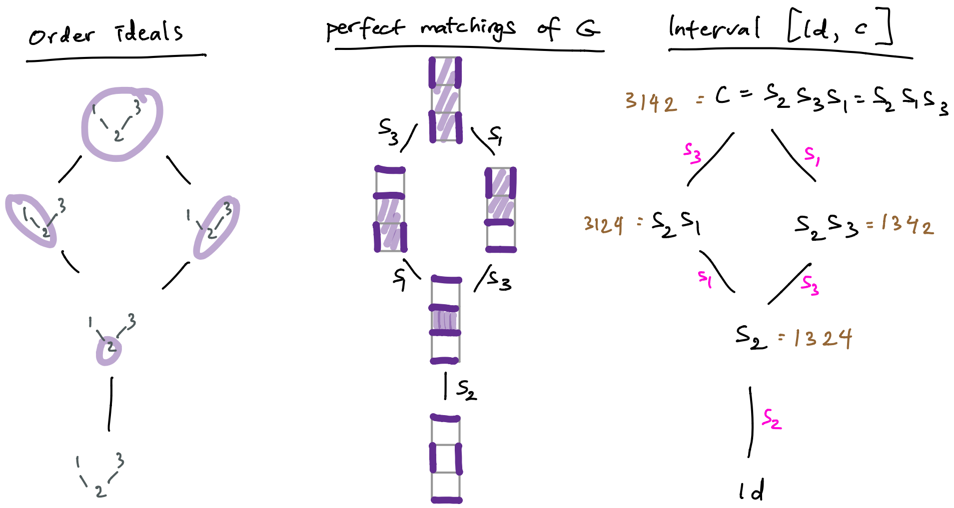
<u>Thm 3.17</u> We have a lattice bijection between $[\mathrm{Id}, c]$ and $\mathcal{L}$.

<u>Example 3</u> Let G be the snake graph $-+$:



Quiver is $Q(G)$ $1 \rightarrow 2 \leftarrow 3$

Coxeter element $c = s_2 s_3 s_1$



In type A, the F-polynomial associated to the cluster variable from c would be $\sum_{w \text{ in } [\mathrm{Id}, c]}$ letters in w $= 1 + y_2 + y_1 y_2 + y_2 y_3 = y_1 y_2 y_3$