Problem I
part 3 b

Conjectures about when $\lambda(\omega) = \lambda^L(\omega)$.

Our Notation of Knuth moves

1st kind K_1^+	2nd kind K_2^+
b c a	c a b
$\uparrow$	$\uparrow$
b a c	a c b

1st kind K_1^-	2nd kind K_2^-
b c a	c a b
$\downarrow$	$\downarrow$
b a c	a c b

Conjecture (5) Checked up to $n=10$

- If Soliton Decomposition$(\pi) \neq$ Row insertion tableau, P, for π,

  then the reading word of P has consecutive terms b_1, a, c, b_2 or b_1, c, a, b_2

  where $a < b_1 < c$ and $a < b_2 < c$

  say "bacb" or "bcab" pattern

  2 1 4 3 (bacb) 2 4 1 3 (bcab)
  3 1 4 2 3 4 1 2

- The contrapositive (is faster to check by computer):
  If the reading word of $P(\pi)$ has no "bacb" and no "bcab" pattern,
  then Soliton Decomposition (π) = $P(\pi)$.

Possible steps to try to prove Conjecture 5

Prove/disprove: If the reading word r of $P(\pi)$ has no "bacb" & no "bcab" pattern,

then π has no "bacb" and no "bcab" pattern.

- Reading word cannot have bacb pattern.

(see Thm 1 below)

Conjecture 6

Checked up to $n=9$ (see Thm 1)

(convention #1)

Let $\pi \in S_n$, $P = RSK(\pi)[0]$, back P = Backward $RSK(\pi)[0]$

Let r be the reading word of P,
and br the backward reading word of back P.

- If there is a path (of Knuth moves) from r to π is such that every edge is a k_1 move or k_2 move (but not both), then Soliton Decomposition $(\pi) = P$.

- If there is a path (of Knuth moves) from br to π is such that every edge is a k_1 move or k_2 move (but not both), then Soliton Decomposition $(\pi) = P$.

Conjecture/Question 7

7a) (Not true!) If every path from r to π and every path from br to π is such that it contains an edge that is both a k_1 and k_2 move, then Soliton Decomposition $(\pi) \neq P$.

small counter example: $\pi = 156234$

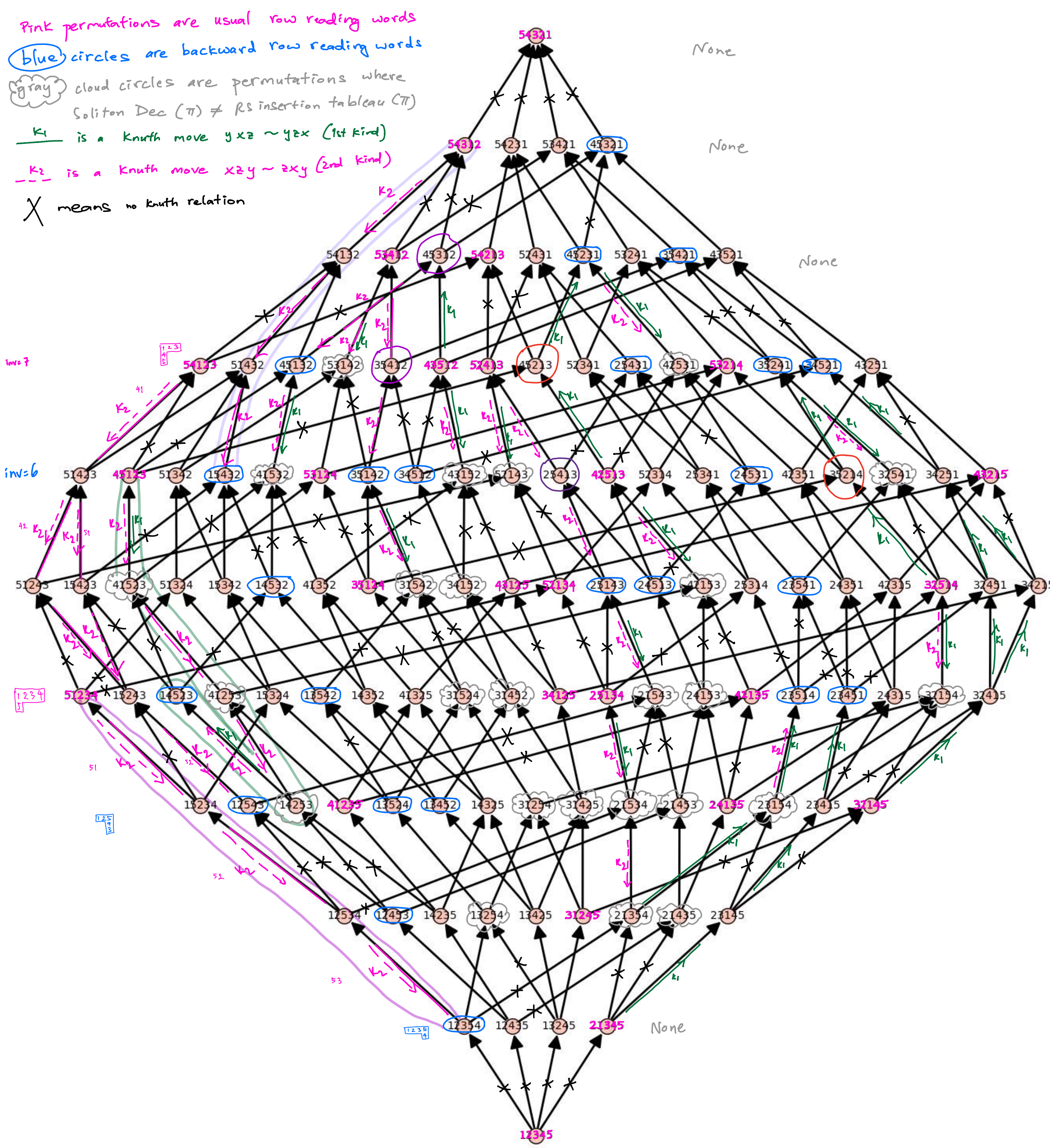
Every path from r to π and every path from br to π contains exactly two edges that are both k_1 and k_2 moves.

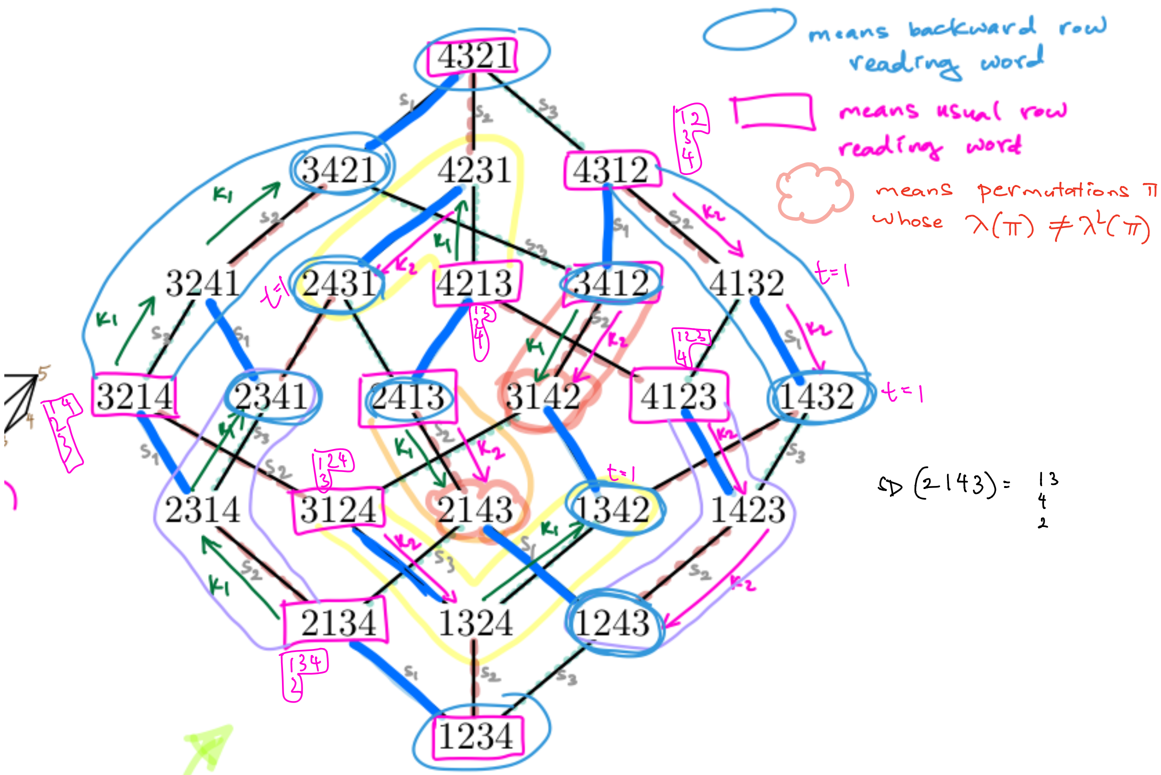
7b) Maybe the following statement is true: If every path from r to π and every path from br to π is such that it contains an ~~even~~ odd number of k_1, k_2 moves, then Soliton Decomposition $(\pi) \neq P$.

7c) If there is a path from r to π which consists of an odd number of k_1, k_2 moves, then Soliton Dec $(\pi) \neq P$.

Pf Applying a k_1-only move or a k_2-only move doesn't change λ^{BBS}.
Applying a k_1, k_2 move changes the height of λ^{BBS} by 1 or -1.

So applying k_1, k_2 moves an odd # of times would result in $\lambda^{BRS}(\pi) \neq \text{sh}\, P$.





(b)

e. b: The right weak order on permutations in S_n.

Recall

• Remark 1

Let $w \in S_n$. Let $\lambda^L(w)$ be the shape of Soliton dec of w.

We know from Exercise 5 (Cor. of [LLPS]) that

$$\lambda^L(w) = \underbrace{\text{the \# of positive parts of } \lambda^L(w)}_{\text{"height" of } \lambda^L} \text{ is } (\text{the \# of descents of } w) + 1.$$

• Remark 2

Let $w \in S_n$, let $P = P(w)$, let r be the row word of P. Then $P = P(r)$.

Let $\lambda(w) = \lambda(r) = \text{sh}\, P$, let $\lambda^L(w)$ be the shape of Soliton dec of w
$\lambda^L(r)$ be the shape of the Soliton dec of r.

We know from Exercise 6 that $\lambda^L(r) = \lambda(r) \left(= \text{sh}\, P\right)$

Note: For some $\lambda^L(w) = \lambda(w) \left(= \text{sh}\, P\right)$, but in general $\lambda^L(w) \trianglelefteq \lambda(w)$

dominance order, see PDF: Problem 1 part 1 (Sec 1.5)

Lemma 1 Let $x, y \in S_n$.

If $x \overset{k}{\longmapsto} \overset{k}{\longrightarrow} \overset{k}{\longrightarrow} \overset{k}{\longrightarrow} \cdots \overset{k}{\longrightarrow} y$ $\quad (k \circ k \circ \cdots \circ k\,(x) = y)$

is a path of $k_1^+, k_1^-, k_2^+, k_2^-$ moves where

each move is either a k_1 move that is not a k_2 move

or a k_2 move that is not a k_1 move,

then the # of pos parts of $\lambda^L(y)$ is equal to

the # of pos parts of $\lambda^L(x)$.

Pf

By Remark 1, for any $w \in S_n$,

# of pos parts of $\lambda^L(w)$ is equal to $\mathrm{desc}(w) + 1$.

So we only need to show $\mathrm{desc}(x) = \mathrm{desc}(y)$.

• Suppose $\ldots ac \ldots \overset{k_1}{\sim} \ldots ca \ldots$ is a k_1 move but not a k_2 move.

Case 1

$k_1^+(\pi) = \ldots bc|ad \ldots$

a is not a descent since $a<c<d$

$\Big\downarrow k_1$

$\pi = \ldots\, b|acd \ldots$ where $c<d$

Case 2

$k_1^+(\pi) = \ldots bc|a|a_1 \ldots$

b is not a descent since $b<c$

$\Big\downarrow k_1$

$\pi = \ldots\, b|ac|a_1 \ldots$ where $a_1 < a$

So $k_1^+(\pi)$ has the same number of descents as π.

• Suppose $\ldots ac \ldots \overset{k_2}{\sim} \ldots ca \ldots$ is a k_2 move but not a k_1 move.

Case 1

$k_2^+(\pi) = \ldots d|c|ab \ldots$

$\Big\downarrow k_2$

$\pi = \ldots d|ac|b \ldots$ where $c<d$

Case 2

$k_2^+(\pi) = \ldots a_1c|ab \ldots$

$\Big\downarrow k_2$

$\pi = \ldots a_1ac|b \ldots$ where $a_1 < a$

So $k_2^+(\pi)$ has the same number of descents as π. $\boxed{\text{Lemma 1}}$

Prop 1 Let r be the row reading word of the tableau P.

Then we cannot apply K_2^+, that is, the Knuth move of the second kind

where we replace $\ldots acb \ldots$ with $cab \ldots$

Pf A perm $r = \ldots a c | b \ldots$ cannot be the reading word of a tableau.

True/False? False

Let r be the reading word of the tableau P.
Then we cannot apply K_1^-, that is, the Knuth move of the 1st kind
where we replace $\ldots bca \ldots$ with $bac \ldots$

is false

Counterexamples: $24|13$, $3|25|14$, $r = 89|247,10|1356$

Prop 2

Let br be the backward reading word of a "backward tableau" bP.

Then we cannot apply K_1^+, the Knuth move of the first kind

where we replace $\ldots bac \ldots$ with $\ldots bca \ldots$

Pf A perm $br = \ldots b|ac \ldots$ cannot be the "backword word" of bP.

Convention # 1

$$P_r \quad \begin{array}{ccc} 7 & 8 & 9 \\ 4 & 5 & 6 \\ & 1 & 3 \\ & & 2 \end{array}$$

or

Convention # 2

$$P_r = \begin{array}{ccc} & & 2 \\ & 1 & 3 \\ 4 & 5 & 6 \\ 7 & 8 & 9 \end{array}$$

$br = 789|456|13|2$

Prop 3

Let $x, y \in S_n$ be such that

$K_1^+(x) = K_2^+(x) = y$.

Then $desc(y) = desc(x) - 1$

Pf $y = \ldots b_1 c | a b_2 \ldots$

$\Big\uparrow K_1^+, K_2^+$

$x = \ldots b_1 | a c | b_2 \ldots$

$a < b_1, b_2 < c$

Prop 4

Let $x, y \in S_n$ be such that

$K_1^-(x) = K_2^-(x) = y$.

Then $desc(y) = desc(x) + 1$

Pf $x = \ldots b_1 c | a b_2 \ldots$

$\Big\downarrow K_1^-, K_2^-$

$y = \ldots b_1 | a c | b_2 \ldots$

$a < b_1, b_2 < c$

Lemma 2(i) Let $w \in S_n$, let $P = P(w)$, let r be the row word of P. (Then $P = P(r)$.)

Let $\lambda(w) = \lambda(r) = \text{sh}\, P$,

let $\lambda^L(w)$ be the shape of soliton dec of w,

$\lambda^L(r)$ be the shape of the soliton dec of r.

Suppose $r \underset{K_1, K_2}{\overset{\text{one Knuth move}}{\sim}} w$ (written $K_1(r) = K_2(r) = w$).

Then: ① $K_1^-(r) = K_2^-(r) = w$

② the # of pos parts of $\lambda^L(w)$ is equal to

the # of pos parts of $\lambda(w) + 1$.

Pf ① By Prop 1, we cannot apply K_2^+ to a row reading word.

② Suppose $r = \ldots b_1\, c \mid a\, b_2 \ldots$ where $\left.\begin{array}{l} a < b_1 < c \\ a < b_2 < c \end{array}\right\}$

Then $K_1^-(r) = K_2^-(r) = w = \ldots b_1 \mid a\ c \mid b_2 \ldots$

So $\text{desc}(w) = \text{desc}(r) + 1$ (or see Prop 4)

Ex: $P = \begin{array}{|cccc|} \hline 1 & 3 & 5 & 6 \\ 2 & 4 & 7, & 10 \\ 8 & 9 & & \\ \hline \end{array}$

$r = 8\,9 \mid 2\,4\,7,10, \mid 1\,3\,5\,6$

By Remark 1, # of pos parts of $\lambda^L(w)$ is equal to $\text{desc}(w) + 1$

and # of pos parts of $\lambda^L(r)$ is equal to $\text{desc}(r) + 1$.

By Remark 2, $\lambda^L(r) = \lambda(r)$ $(= \text{sh}\, P)$ $\boxed{\text{Lemma}}$

Lemma 2(ii) Let $w \in S_n$, let bP = backward RS insertion tableau of w.

Let br be the backward row reading word of bP.

Let $\lambda(w) = \lambda(br) = \text{sh}\ bP$.

Let $\lambda^L(w)$ be the shape of the backward soliton decomposition of w

(which is known to be the same as the shape of the forward soliton dec of w).

Let $\lambda^L(br)$ be the shape of the backward soliton decomposition of br

(which is known to be the same as the shape of the forward soliton dec of br).

Suppose $br \underset{K_1, K_2}{\overset{\text{one Knuth move}}{\sim}} w$ (written $K_1(br) = K_2(br) = w$).

Then: ① $K_1^-(br) = K_2^-(br) = w$

② the # of pos parts of $\lambda^L(w)$ is equal to

the # of pos parts of $\lambda(w) + 1$.

Pf ① By Prop 2, we cannot apply K_1^+ to a backward row reading word.

② Same proof as in Lemma 2(i)

Ex:

Convention #2

$\begin{array}{cccc} 2 & 8 & 9 & 10 \\ 1 & 4 & 5 & 7 \\ & & 3 & 6 \end{array}$

Convention #2

$\begin{array}{cccc} & & 3 & 6 \\ 1 & 4 & 5 & 7 \\ 2 & 8 & 9 & 10 \end{array}$

$br = 2\,8\,9,10, \mid 1\,4\,5\,7 \mid 3\,6$

Standard Def

Let $\pi \in S_n$

Let $i(\pi) := \max \{ k \in [n] \mid \pi_{i_1} < \dots < \pi_{i_k}$ for some $i_1 < \dots < i_k \}$
denote the size of a longest increasing subsequence of π.

Let $u = (u_1, u_2, \dots)$ be a finite sequence of numbers.

Let $i(u) := \max \{ k \geqslant 1 \mid u_{i_1} < \dots < u_{i_k}$ for some $i_1 < \dots < i_k \}$
denote the size of a longest increasing subsequence of π.

Choose our notation

Let $I_0 := 0$

Let $$I_k := \max_{\pi = u_1 | u_2 | \cdots | u_k} \left(i(u_1) + i(u_2) + \dots + i(u_k) \right)$$

where the maximum is taken over ways of writing π as a concatenation $u_1 | u_2 | \dots | u_k$ of consecutive subsequences of π.
[LLPS] uses the term "non-interlacing"

Let $\lambda_k = \lambda_k^{BBS}$ or λ_k^L $:= I_k - I_{k-1}$ for $k \geqslant 1$

Let $\lambda = \lambda^{BBS}$ or λ^L $= (\lambda_1, \lambda_2, \lambda_3, \dots)$

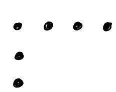
E.g.
$$\begin{array}{lcccccc} & 1 & 2 & 3 & 4 & 5 & 6 \\ \pi = & 1 & 5 & 2 & 3 & 6 & 4 \end{array}$$

$I_1 = i(\pi) = 4$

$I_2 = 5$ (witnessed by $15|2364$, $15236|4$, etc)

$I_3 = 6$

$\lambda = (4, 1, 1)$



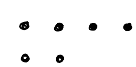
Standard Def

Let $i_k(\pi)$ be the length of π's longest k-increasing subsequence (Sagan Sec 3.5)

$\lambda^{RS} :=$ shape of RS tableau

E.g. $i_1 = I_1 = 4$
$i_2 = 6$
$i_3 = 6$

$\lambda^{RS}(\pi) =$



Note: $I_k \leq i_k$
usual comparison of numbers

So $\lambda^{BBS} \leq \lambda^{RS}$
Dominance ordering on integer partitions

Standard Def

- Let $\pi \in S_n$. We say that $i \in [n-1]$ is a descent of π if $\pi_i > \pi_{i+1}$.

  The descent set of π is $\mathrm{Des}(\pi) := \{ i \in [n-1] \mid \pi_i > \pi_{i+1} \}$.
  or $\mathrm{Desc}(\pi)$ if you prefer

  The number of descents of π is denoted $\mathrm{des}(\pi)$. or $\mathrm{desc}(\pi)$

- Let $u = (u_1, u_2, \ldots)$ be a sequence of numbers.

  We say that i is a descent of u if $u_i > u_{i+1}$.

  The descent set of is $\mathrm{Des}(u) := \{ i \in \mathbb{Z}_{\geq 1} \mid u_i > u_{i+1} \}$.
  or $\mathrm{Desc}(u)$ if you prefer

  If u is finite, the number of descents of u is denoted $\mathrm{des}(u)$.
  or $\mathrm{desc}(u)$

Choose our convention

des* is like in Lewis' blog

Let $\mathrm{des}^*(u) := \begin{cases} 0 & \text{if } u \text{ is the empty sequence} \\ 1 + \mathrm{des}(u) & \text{otherwise} \end{cases}$

or $\mathrm{ND}(u)$
like NA in [LLPS],
or $\mathrm{ndes}(u)$

Let $D_0 := 0$,
$$D_k := \max_{\pi = u_1 \sqcup u_2 \sqcup \ldots \sqcup u_k} \left(\mathrm{des}^*(u_1) + \ldots + \mathrm{des}^*(u_k) \right) \quad \text{for } k \geq 1$$
where the maximum is taken over disjoint subsequences u_i of π.

Let $M_k = M_k^{BBS} := D_k - D_{k-1}$ for $k \geq 1$
or M_k^L

Let $M = M^{BBS} = (\lambda_1, \lambda_2, \lambda_3, \ldots)$
or M^L

E.g.
$$\begin{array}{ccccccc} & 1 & 2 & 3 & 4 & 5 & 6 \\ \pi = & 1 & 5 & 2 & 3 & 6 & 4 \end{array}$$

$D_1 = \mathrm{des}^*(\pi) = |\{2, 5\}| + 1 = 3$

$D_2 = \mathrm{des}^*(1264) + \mathrm{des}^*(53) = 2 + 2 = 4$

$D_3 = 5$

$D_4 = 6$ $\qquad M = (3, 1, 1, 1)$ $\quad \vdots$

Standard Def

Let $d_k(\pi)$ be the length of π's longest k-decreasing subsequence (Sagan Sec 3.5)

$M^{RS} :=$ shape of $(\text{RS tableau})^{\text{transpose}}$

$d_1 = 2$
$d_2 = 4$
$d_3 = 5$
$d_4 = 6$

$M^{RS}(\pi) = (2, 2, 1, 1)$ $\vdots$

Note: $D_k \geq d_k$
usual comparison of numbers

So $M^{BBS} \geq M^{RS}$
Dominance ordering on integer partitions

Thm 1

$\pi, \omega \in S_n$

If π and ω differ by one Knuth move that is not both a Knuth move of the 1st & 2nd kind,

then $D_k(\pi) = D_k(\omega)$ $\forall$ k.

Proof of Thm 1

Case 1 (i)

$y = \kappa_1^+(\pi) = \ldots bcad \ldots$ $(c < d)$

$x = \pi = \ldots bacd \ldots$

Case 1 (ii)

$y = \kappa_1^+(\pi) = \ldots bca\alpha \ldots \alpha < a$

$x = \pi = \ldots bac\alpha \ldots$

Case 2 (i)

$y = \kappa_2^+(\pi) = \ldots dcab \ldots$ $(c < d)$

$x = \pi = \ldots dacb \ldots$

Case 2 (ii)

$y = \kappa_2^+(\pi) = \ldots \alpha cab \ldots$ $(\alpha < a)$

$x = \pi = \ldots \alpha acb \ldots$

Suppose that $u_1, u_2, \cdots, u_k$ are disjoint subsequences of y

s.t. $D_k(y) = des^*(u_1) + \ldots + des^*(u_k)$.

(We will now show $D_k(y) \le D_k(x)$ by producing disjoint subsequences $u_1', u_2', \ldots, u_k'$ of x where $\sum_{i=1}^{k} des^*(u_i) \le \sum_{i=1}^{k} des^*(u_i')$.)

See Fig 1 for 15 different cases.

- First, suppose c and a are in distinct subsequences. (In the figure, all but ①–⑤ fit into this case.)
  Then set $u_i' := u_i$ for all $i \in [k]$.
  Since $des^*(u_i') = des^*(u_i)$ for all $i \in [k]$, we have $D_k(y) \le D_k(x)$.

- Next, suppose b, c, and a are in the same subsequence u_j of y. (In the fig, see ⑤ and ④).

  Define u_j' to be the subsequence of x which is obtained by swapping c, a with a, c.
  Define $u_i' := u_i$ for all $i \in [k] - \{j\}$. Then

  $des(u_j) = |Des((c \ldots, b, c, a, \ldots))| \le |Des((c \ldots, b, a, c, \ldots))| = des(u_j')$, so $D_k(y) \le D_k(x)$

  may or may not be a descent

- Finally, suppose c and a are in the same subsequence, say (wlog), u_1, of y

  and b is in a different subsequence, say, u_2, of y. (In fig, see ③, ②, ①)

  Write u_1 as a concatenation

  $$u_1 = \overbrace{(\ldots, c)}^{u_1^{I} :=} \sqcup \overbrace{(a, \ldots)}^{u_1^{II} :=}$$

  of two subsequences u_1^{I} (ending in c) and u_1^{II} (starting in a) of y.

  Write u_2 as a concatenation

  $$u_2 = \overbrace{(\ldots, b)}^{u_2^{I} :=} \sqcup \overbrace{(\ldots)}^{u_2^{II} :=}$$

  may or may not be a descent

  of two subsequences u_2^{I} (ending in b) and u_2^{II} of y.
  Note that, if u_2 ends in b, then u_2^{II} is empty.

  $$\text{Define } u_1' := \overbrace{(\ldots, b)}^{u_2^{I}} \sqcup \overbrace{(a, \ldots)}^{u_1^{II}},$$

  $$u_2' := \overbrace{(\ldots, c)}^{u_1^{I}} \sqcup \overbrace{(\ldots)}^{u_2^{II}},$$

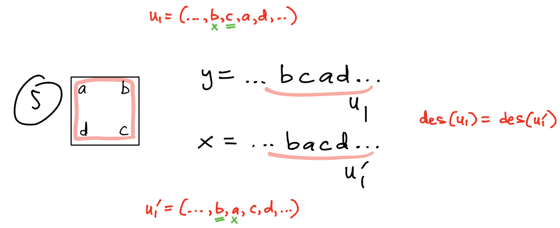
  [if b contributes to $des(u_2)$, then c contributes to $des(u_2')$]

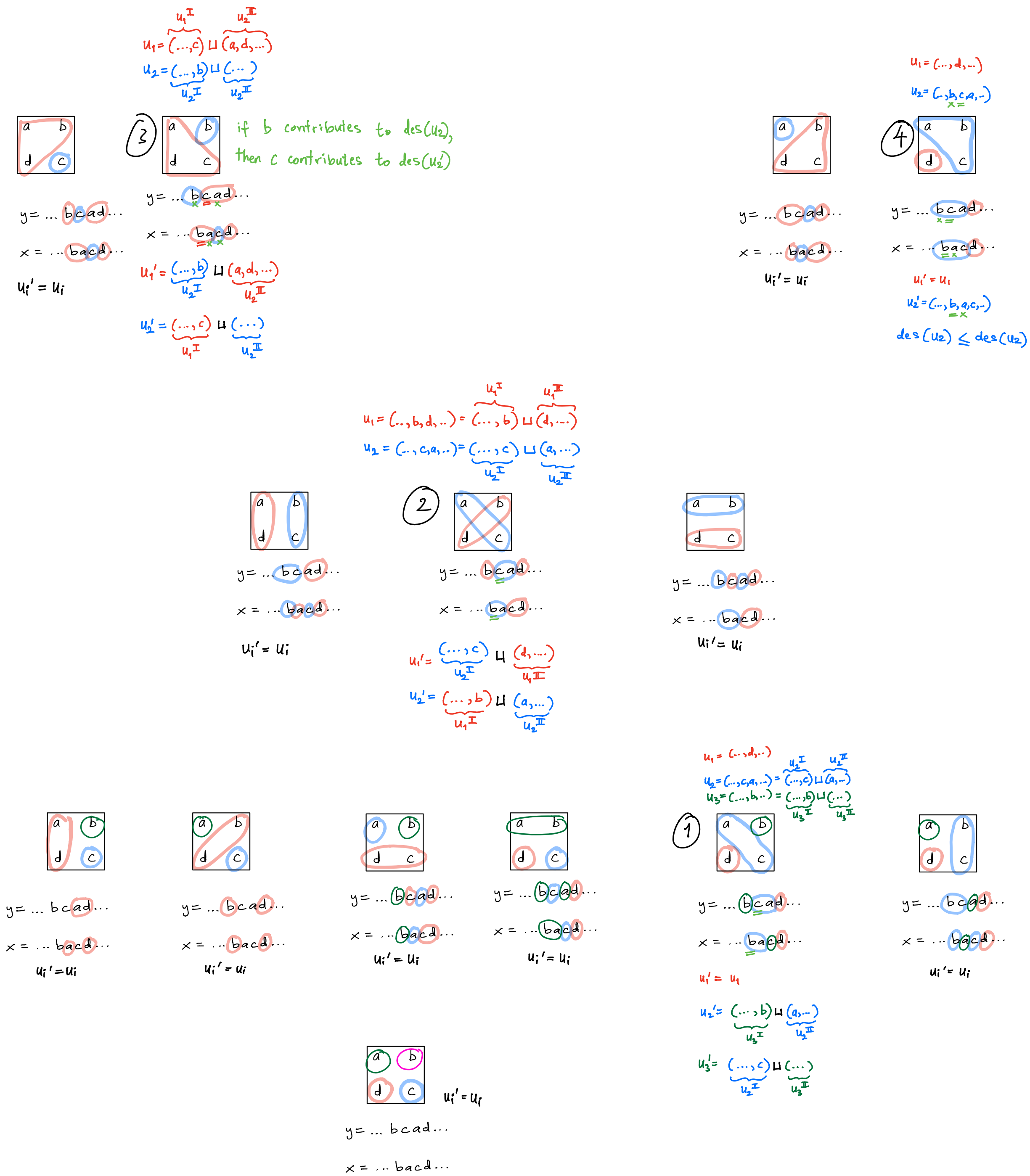
  and $u_i' := u_i$ for all $i \in [k] - \{1, 2\}$.
  Then $des(u_1) + des(u_2) \le des(u_1') + des(u_2')$

Given $y = u_1 \sqcup \ldots \sqcup u_k$,

construct $x = u_1' \sqcup \ldots \sqcup u_k'$.

Fig 1





Case 1(i) (cont)

$y = k_1^+(\pi) = \ldots bcad \ldots$ $(c < d)$

$x = \pi = \ldots bacd \ldots$

Suppose that $v_1, v_2, \cdots, v_k$ are disjoint subsequences of x

s.t. $D_k(x) = \mathrm{des}^*(v_1) + \ldots + \mathrm{des}^*(v_k)$.

(We will now show $D_k(x) \le D_k(y)$ by producing disjoint subsequences $v_1', v_2', \ldots, v_k'$ of y where $\sum_{i=1}^{k} \mathrm{des}^*(v_i) \le \sum_{i=1}^{k} \mathrm{des}^*(v_i')$.)

See Fig 2 for 15 different cases.

- First, suppose c and a are in distinct subsequences. (In Fig 2, all but ①–⑤ fit into this case.)
  Then set $v_i' := v_i$ for all $i \in [k]$.
  Since $\mathrm{des}^*(v_i') = \mathrm{des}^*(v_i)$ for all $i \in [k]$, we have $D_k(x) \le D_k(y)$.

- Next, suppose b, a, c, d are in the same subsequence v_j of x. (In the fig 2, see ⑤).

  Define v_j' to be the subsequence of x which is obtained by swapping b,a,c,d with b,c,a,d.
  Define $v_i' := v_i$ for all $i \in [k] - \{j\}$.

Then $\mathrm{des}(v_j) = |\mathrm{Des}((\ldots, \underset{=}{b}, \underset{\times}{a}, \underset{\times}{c}, d, \ldots))| = |\mathrm{Des}((\ldots, \underset{\times}{b}, \underset{=}{c}, \underset{\times}{a}, d, \ldots))| = \mathrm{des}(v_j')$, so $D_k(x) \le D_k(y)$.

- Suppose a, c, d are in the same subsequence, say, v_1, of x, and b is in a different subsequence, say, v_2 of x. (See Fig 2, ③)

  Write $v_1 = (\ldots, a, c, d, \ldots)$ as a concatenation

  $$v_1 = \underbrace{(\cdots, a)}_{v_1^{\mathrm{I}}} \sqcup (c) \sqcup \underbrace{(d, \ldots)}_{v_1^{\mathrm{II}}}$$

  of three subsequences v_1^{I} (ending with a),
  the one-element sequence (c),
  and v_1^{II} (starting with d) of x.

  Write $v_2 = (\ldots, b, \ldots)$ as a concatenation

  $$v_2 = \underbrace{(\ldots, b)}_{v_2^{\mathrm{I}}} \sqcup \underbrace{(\ldots)}_{v_2^{\mathrm{II}}}$$

  of two subsequences v_2^{I} (ending with b) and v_2^{II} of x.
  Note that, if v_2 ends in b, then v_2^{II} is empty.

  Define

  $$v_1' := \underbrace{(\cdots, a)}_{v_1^{\mathrm{I}}} \sqcup \underbrace{(d, \ldots)}_{v_1^{\mathrm{II}}}$$

  $$v_2' := \underbrace{(\ldots, b)}_{v_2^{\mathrm{I}}} \sqcup (c) \sqcup \underbrace{(\ldots)}_{v_2^{\mathrm{II}}}$$

  Note that c doesn't contribute to $\mathrm{des}(v_1)$, and b doesn't contribute to $\mathrm{des}(v_2)$.
  If b contributes to $\mathrm{des}(v_2)$, then c contributes to $\mathrm{des}(v_2')$. So
  $\mathrm{des}^*(v_1) + \mathrm{des}^*(v_2) = \mathrm{des}^*(v_1') + \mathrm{des}^*(v_2')$.
  Hence $D_k(x) \le D_k(y)$.

- Finally, suppose c and a are in the same subsequence, say (wlog), v_1, of x and d is in a different subsequence, say, v_2, of x. (In fig 2, see ④, ②, ①)

Write v_1 as a concatenation

$$v_1 = \overbrace{(\ldots, a)}^{v_1^{I} :=} \sqcup \overbrace{(c, \ldots)}^{v_1^{II} :=}$$

of two subsequences v_1^{I} (starting in a) and v_1^{II} (ending in c) of x.

Write v_2 as a concatenation

$$v_2 = \overbrace{(\ldots\ )}^{v_2^{I} :=} \sqcup \overbrace{(d, \ldots)}^{v_2^{II} :=}$$

may or may not be a descent

of two subsequences v_2^{I} and v_2^{II} (starting with d) of x.

Note that, if v_2 starts w/ d, then v_2^{I} is empty.

$$\text{Define } v_1' := \overbrace{(\ \ldots\ )}^{v_2^{I}} \sqcup \overbrace{(c, \ldots)}^{v_1^{II}},$$

$$v_2' := \overbrace{(\ldots, a)}^{v_1^{I}} \sqcup \overbrace{(d, \ldots)}^{v_2^{II}},$$

and $v_i' := v_i$ for all $i \in [k] - \{1, 2\}$.

Then $\operatorname{des}(v_1) + \operatorname{des}(v_2) \leq \operatorname{des}(v_1') + \operatorname{des}(v_2')$

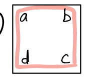
since, if the element to the left of d in v_2 contributes to $\operatorname{des}(v_2)$, this element contributes to $\operatorname{des}(v_1')$.

Given $x = v_1 \sqcup \dots \sqcup v_k$,

construct $y = v_1' \sqcup \dots \sqcup v_k'$.

Fig 2

$v_1' = (\dots, b, c, a, d, \dots)$

(5) 

$y = \dots \underbrace{bcad}_{u_1} \dots$ $des(v_1) = des(v_1')$

$x = \dots \underbrace{bacd}_{u_1'} \dots$

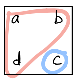
$v_1 = (\dots, b, a, c, d, \dots)$

$v_1' = (\dots, a) \sqcup (d, \dots)$, with $v_1^{I} = (\dots, a)$ and $v_2^{II} = (d, \dots)$

$v_2' = (\dots, b) \sqcup (c) \sqcup (\dots)$, with $v_2^{I} = (\dots, b)$ and $v_2^{II} = (\dots)$

$v_1' = \underbrace{(\dots)}_{v_1^{I}} \sqcup \underbrace{(c, \dots)}_{v_2^{II}}$

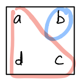
$v_2' = \underbrace{(\dots, b, a)}_{v_2^{I}} \sqcup \underbrace{(d, \dots)}_{v_1^{II}}$



$y = \dots bcad \dots$

$x = \dots bacd \dots$

$v_i' := v_i$

(3) 

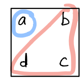
if b contributes to $des(v_2)$, then c contributes to $des(v_2')$

$y = \dots bcad \dots$

$x = \dots bacd \dots$

$v_1 = \underbrace{(\dots, a)}_{v_1^{I}} \sqcup (c) \sqcup \underbrace{(d, \dots)}_{v_1^{II}}$

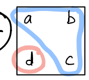
$v_2 = \underbrace{(\dots, b)}_{v_2^{I}} \sqcup \underbrace{(\dots)}_{v_2^{II}}$



$y = \dots bcad \dots$

$x = \dots bacd \dots$

$v_i' = v_i$

(4) 

$y = \dots bcad \dots$

$x = \dots bacd \dots$

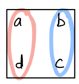
$v_1 = \underbrace{(\dots)}_{v_1^{I}} \sqcup \underbrace{(d, \dots)}_{v_1^{II}}$

$v_2 = \underbrace{(\dots, b, a)}_{v_2^{I}} \sqcup \underbrace{(c, \dots)}_{v_2^{II}}$

$des(v_1) + des(v_2) = des(v_1') + des(v_2')$

$v_1' = \underbrace{(\dots, b)}_{v_1^{I}} \sqcup \underbrace{(c, \dots)}_{v_2^{II}}$

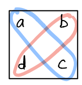
$v_2' = \underbrace{(\dots, a)}_{v_2^{I}} \sqcup \underbrace{(d, \dots)}_{v_1^{II}}$



$y = \dots bcad \dots$

$x = \dots bacd \dots$

$v_i' = v_i$

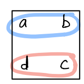
(2) 

$y = \dots bcad \dots$

$x = \dots bacd \dots$

$v_1 = \underbrace{(\dots, b)}_{v_1^{I}} \sqcup \underbrace{(d, \dots)}_{v_1^{II}}$

$v_2 = \underbrace{(\dots, a)}_{v_2^{I}} \sqcup \underbrace{(c, \dots)}_{v_2^{II}}$



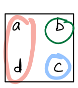
$y = \dots bcad \dots$

$x = \dots bacd \dots$

$v_i' = v_i$

$v_1' = \underbrace{(\dots, a)}_{v_2^{I}} \sqcup \underbrace{(d, \dots)}_{v_1^{II}}$

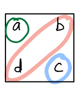
$v_2' = \underbrace{(\dots)}_{v_1^{I}} \sqcup \underbrace{(c, \dots)}_{v_2^{II}}$



$y = \dots bcad \dots$

$x = \dots bacd \dots$

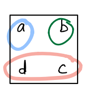
$v_i' = v_i$



$y = \dots bcad \dots$

$x = \dots bacd \dots$

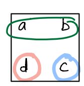
$v_i' = v_i$



$y = \dots bcad \dots$

$x = \dots bacd \dots$

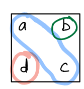
$v_i' = v_i$



$y = \dots bcad \dots$

$x = \dots bacd \dots$

$v_i' = v_i$

(1) 

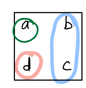
$y = \dots bcad \dots$

$x = \dots bacd \dots$

$v_1 = \underbrace{(\dots)}_{v_1^{I}} \sqcup \underbrace{(d, \dots)}_{v_1^{II}}$

$v_2 = \underbrace{(\dots, a)}_{v_2^{I}} \sqcup \underbrace{(c, \dots)}_{v_2^{II}}$

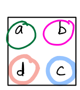
if the last elt of v_1^{I} contributes to $des(v_1)$, then this same elt contributes to $des(v_2')$ since $c < d$



$y = \dots bcad \dots$

$x = \dots bacd \dots$

$v_i' = v_i$



$v_i' = v_i$

$y = \dots bcad \dots$

$x = \dots bacd \dots$

end of case 1(i) of Thm 1