# PROBLEM 1 Notes, part 2

Ref:
- Textbook "Enumerative Combinatorics Vol 2" by Richard Stanley Sec 7.13 "Symmetry of the RSK Algorithm" p. 324
- Sage Math documentation (Google "Sage Math" + a math concept)
- Blog post "Localized version of Greene's thm" by Joel B. Lewis

( Last update : Wed, June 3, 2020 )

**Sage Note:**

To change the default permutation multiplication to be right to left:

```
sage: Permutations.options.mult = 'r2l'
```

## 1.6 Chains, antichains, Hasse diagrams

<u>Def</u> Let $(P, \leq)$ be a partially ordered set (poset). See Sec 1.5.

- A chain (or total order) is a subset of $P$ which contains no two incomparable elements
- An antichain is a subset of $P$ which contains no two comparable elements.
- A chain (resp. antichain) $S$ is maximal if no elt can be added to $S$ w/o destroying the chain property of $S$ (resp. antichain)

- We say $x$ is covered by $y$ (also, $y$ covers $x$).

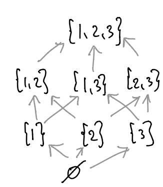
  when $x \lneq y$ and there is no point $z \in P$ for which $x \lneq z \lneq y$.

- The Hasse diagram of $P$ is a graph whose vertices represent the elements of $P$. If $x \lneq y$ in $P$, draw $y$ above $x$.

  If $y$ covers $x$, then there is an edge between $x$ and $y$. (sometimes we put a directed edge from $x$ to $y$).

### Hasse diagrams Examples

(i) Boolean algebra for subsets of $[n]$, ordered by inclusion. Here $n=3$.




sage: P=posets.BooleanLattice(3)
sage: P
sage: P.cover_relations()
sage: len(_) ← underscore
10
sage: P.width() # size of a largest antichain
3

(ii) The right (as opposed to left) weak order on the symmetric group $S_n$. (Relevant for REU PROBLEM II)

<u>Def</u> • If $\pi \in S_n$ and $i<j$, $\pi(i) > \pi(j)$, then the pair $(i,j)$ is an inversion of $\pi$.

sage: pi = Permutation([4,3,2,1])
sage: pi.inversions()
sage: len(_) # every pair $(i,j)$, $i<j$
6

• In the weak order on the set $S_n$,

$x$ is covered by $y$ iff $x\,(i, i+1) = y$, (multiply R 2 L)

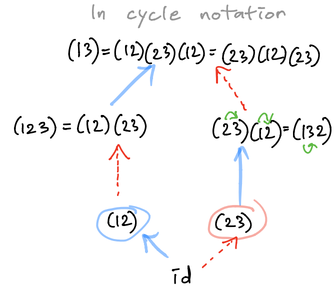
and (the # of inversions of $x$)+1 = the # of inversions of $y$.

Here $n=3$.

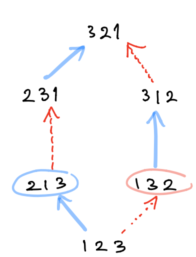
$\binom{n}{k} :=$ # of $k$-element subsets of $[n]$ $= \frac{n!}{k!\,(n-k)!}$

Side Note: The Hasse diagram of the weak order looks like the Cayley diagram of the group $S_n$, with generators $(1,2), (2,3), \dots, (n-1,n)$. (cycle notation)

In cycle notation




In one line notation




$\# inv = \binom{n}{2} = 3$

$\# inv = \binom{n}{2} - 1 = 2$

$\# inv = 1$

$\# inv = 0$

sage: S3 = posets.SymmetricGroupWeakOrderPoset(3)
sage: view(S3)
sage: S3.maximal_chains()
sage: S3.maximal_antichains()
sage: S3.greene_shape() # explained at end of PDF
[4,2]

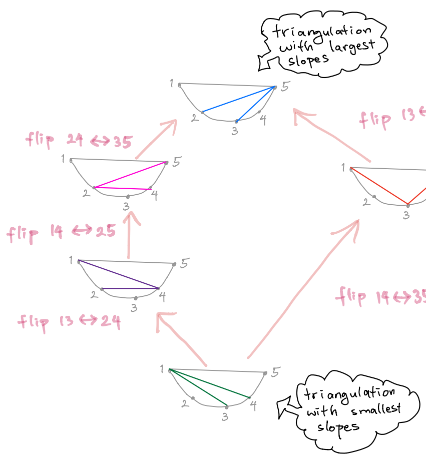
## (iii) Tamari lattice (on 200+ families of Catalan objects)

Relevant for REU PROBLEM II

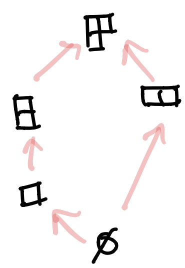
a. The Tamari lattice on triangulations of a polygon.

Say $T_1$ is covered by $T_2$ if $T_2$ is the result of removing a diagonal of $T_1$ and replacing it with another diagonal of larger slope (equivalently, a diagonal with higher endpoints).

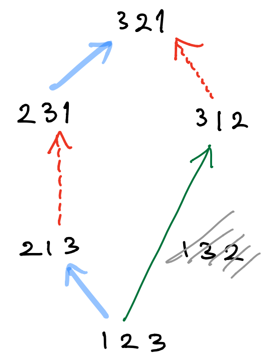
Here the polygon is a pentagon.




b. The Tamari lattice on Young diagrams which fit into the staircase $(r, r-1, r-2, \dots, 1)$ partition

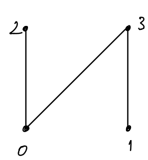



c. The Tamari lattice on permutations in $S_n$ which avoid the pattern 132 (low, high, middle), meaning the one line notation has no subsequence of the form $a, c, b$, where $a < b < c$.



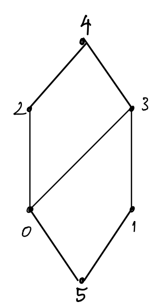

```
sage: P = posets.TamariLattice(3)
sage: len(P.maximal_chains())
sage: view(P)
sage: P.greene_shape()   # explained at end of PDF
[4, 1]
sage: P.width()   # size of a largest antichain
2
```

## (iv) Other poset examples

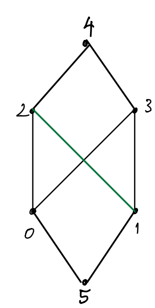



cover relations

```
sage: D = DiGraph([(0,2), (0,3), (1,3)])
sage: P1 = Poset(D)
sage: view(P1)
sage: P1.greene_shape()
[2,2]
sage: P1.maximal_chains()
sage: P1.is_lattice()
False
```




```
sage: D = DiGraph([(5,0),(5,1),(1,3),(0,3),
                   (0,2),(3,4),(2,4)])
sage: P2 = Poset(D)
sage: view(P2)
sage: P2.greene_shape()
[4,2]
sage: P2.maximal_antichains()
sage: P2.is_lattice()
True
```




```
sage: D = DiGraph([(5,0),(5,1),(1,3),(0,3),
                   (1,2),(0,2),(3,4),(2,4)])
sage: P3 = Poset(D)
sage: view(P )
sage: P3.greene_shape()
[4,2]
sage: P3.maximal_antichains()
sage: P3.is_lattice()
False
```

# 1.7 Inversion Poset

(Ref: Textbook "Enumerative Combinatorics Vol 2"
Sec 7.13 "Symmetry of the RSK Algorithm" p. 324)

__Def__ Let $\pi = \begin{pmatrix} 1 & 2 & 3 & \cdots & n \\ \pi_1 & \pi_2 & \pi_3 & \cdots & \pi_n \end{pmatrix} \in S_n$ written in two line notation.

Define the inversion poset (or permutation poset) $I(\pi)$ as follows.

often used to mean different things in other situations.

- The vertices / elements of $I(\pi)$ are the columns $\binom{i}{\pi_i}$ or tuple $(i, \pi_i)$ for all $i \in [n]$.
- Define $\binom{i}{\pi_i} \leq \binom{j}{\pi_j}$ if $i \leq j$ and $\pi_i \leq \pi_j$

__Example:__ Let $\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 3 & 7 & 6 & 1 & 4 & 2 & 5 \end{pmatrix}$. Then the vertices $\binom{1}{3}$ and $\binom{4}{1}$ are not comparable, $\binom{5}{4}$ and $\binom{6}{2}$ are not comparable, etc.

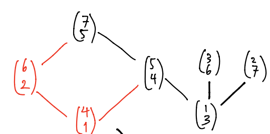
The vertex $\binom{1}{3}$ is only comparable to $\binom{2}{7}$, $\binom{3}{6}$, $\binom{5}{4}$, and $\binom{7}{5}$.

The vertex $\binom{2}{7}$ is a maximal element, since $n = 7$.

The vertex $\binom{4}{1}$ is smaller than $\binom{i}{\pi_i}$ for all $i > 4$

```
sage: w= Permutation([3, 7, 6, 1, 4, 2, 5])
sage : PP = w. permutation_poset()
sage : PP. cover_relations()
```

Hasse diagram of $I(\pi)$



Compute $\lambda$ and $\mu$ for $\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 3 & 7 & 6 & 1 & 4 & 2 & 5 \end{pmatrix}$.

$A_0 = 0$

$A_1 =$ size of a longest increasing subsequence $= 3$ (e.g. 1,2,5)

$A_2 =$ size of largest union of two increasing subsequences

several possible decompositions

$= |\{3\ 7\ 6\ 1\ 4\ 2\ 5,\ 3\ 7\ 6\ 1\ 4\ 2\ 5\}|$

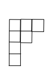
$= |\{3\ 7\ 6\ 1\ 4\ 2\ 5,\ 3\ 7\ 6\ 1\ 4\ 2\ 5\}| = 5$

$A_3 = |\{3\ 7\ 6\ 1\ 4\ 2\ 5,\ 3\ 7\ 6\ 1\ 4\ 2\ 5,\ 3\ 7\ 6\ 1\ 4\ 2\ 5\}| = 6$

several possible decompositions

$A_4 = |\{3\ 7\ 6\ 1\ 4\ 2\ 5,\ 3\ 7\ 6\ 1\ 4\ 2\ 5,\ 3\ 7\ 6\ 1\ 4\ 2\ 5,\ 3\ 7\ 6\ 1\ 4\ 2\ 5\}| = 7$

$\therefore \lambda = (3, 2, 1, 1)$



$D_0 = 0$

$D_1 =$ size of a longest decreasing subsequence

$= 4$ (witnessed uniquely by 7, 6, 4, 2)

$D_2 =$ size of largest union of two decreasing subsequences

$= |\{3\ 7\ 6\ 1\ 4\ 2\ 5,\ 3\ 7\ 6\ 1\ 4\ 2\ 5\}|$ unique decomposition

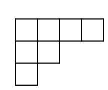
$= 6$

$D_3 = |\{3\ 7\ 6\ 1\ 4\ 2\ 5,\ 3\ 7\ 6\ 1\ 4\ 2\ 5,\ 3\ 7\ 6\ 1\ 4\ 2\ 5\}|$

several possible decompositions

$= 7$

$\mu = (4, 2, 1)$



__Rem__

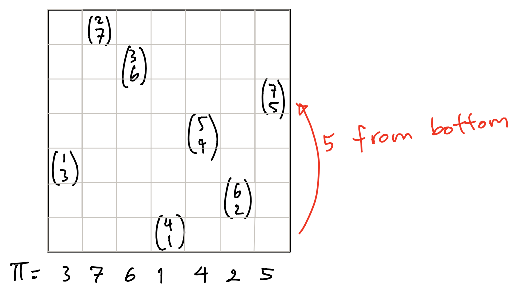
1) $\overset{4}{1}, \overset{6}{2}, \overset{7}{5}$ is a longest chain
2) $\overset{1}{3}, \overset{5}{4}$ and $\overset{4}{1}, \overset{6}{2}, \overset{7}{5}$ give a largest union of 2 chains
3) $\overset{1}{3}, \overset{2}{7}$ and $\overset{4}{1}, \overset{6}{2}, \overset{7}{5}$ and $\overset{3}{6}$ give a largest union of 3 chains.
4) $\overset{1}{3}, \overset{5}{4}, \overset{7}{5}$ and $\overset{4}{1}, \overset{6}{2}$ and $\overset{3}{6}$ and $\overset{2}{7}$ give a largest union of 4 chains.

1.) $\{\overset{2}{7}, \overset{3}{6}, \overset{5}{4}, \overset{6}{2}\}$ is the largest antichain
2.) $\{\overset{2}{7}, \overset{3}{6}, \overset{5}{4}, \overset{6}{2}\}$ and $\{\overset{1}{3}, \overset{4}{1}\}$ give the largest union of 2 antichains
3.) $\{\overset{2}{7}, \overset{3}{6}, \overset{7}{5}\}$ and $\{\overset{1}{3}, \overset{4}{1}\}$ and $\{\overset{5}{4}, \overset{6}{2}\}$ give the largest union of 3 antichains.

## An equivalent def of the inversion poset $I(\pi)$:

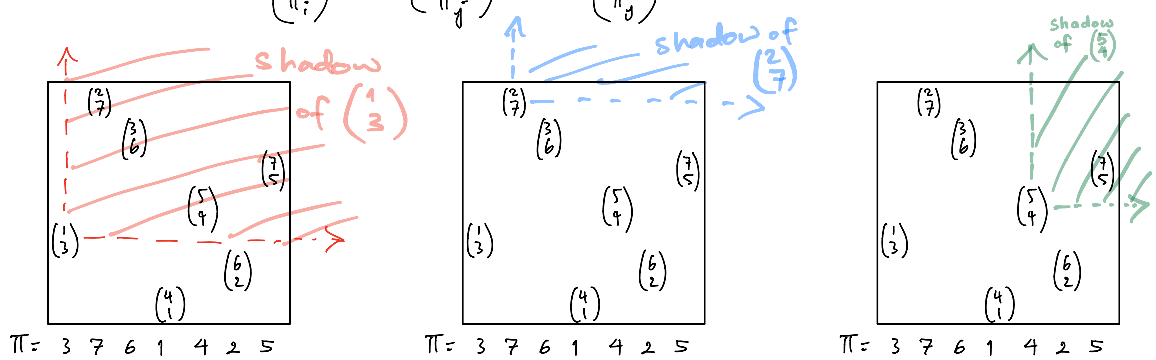
Construct an $n \times n$ permutation matrix, with $\binom{i}{\pi_i}$ in the $\pi_i$ th square from the bottom of column $i$:

E.g. $\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 3 & 7 & 6 & 1 & 4 & 2 & 5 \end{pmatrix}$. from the previous example.

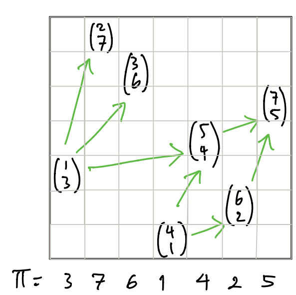


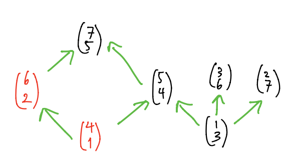

- These are the $n$ points of the poset $I(\pi)$.
- Draw a line pointing north and a line pointing east from each point $\binom{i}{\pi_i}$. These two lines determine a one quarter region of the plane. For now, call this region the shadow of $\binom{i}{\pi_i}$.

Then define $\binom{i}{\pi_i} \leq \binom{j}{\pi_j}$ if $\binom{j}{\pi_j}$ is inside the shadow of $\binom{i}{\pi_i}$.




Get the same poset $I(\pi)$ as on the last page



## 1.8 Greene-Kleitman invariant / partition

Thm (Greene, Kleitman, "The structure of Sperner k-families" 1976)

- $A_k$ is the size of a largest union of $k$ chains in the inversion poset $I(\pi)$.
- $D_k$ is the size of a largest union of $k$ antichains in the inversion poset $I(\pi)$.

(In fact, we can replace poset "$I(\pi)$" with any poset $P$, define $A_k$ to be the size of a largest union of $k$ chains in $P$, define $D_k$ to be the size of a largest union of $k$ antichains in $P$, and get two conjugate partitions.

Here $\lambda$ is often called the Greene-Kleitman invariant/partition of $P$)

```
sage: for n in range(2,14):
... :        Dom = posets.IntegerPartitionsDominanceOrder(n)
... :        print(n)
... :        Dom.greene_shape()
```

```
sage: w= Permutation([3,1,5,2,7,6,8,4])   # Mu L = [3,2,2,1]
sage : PP = w.permutation_poset()
sage : PP.cover_relations()
sage: PP.greene_shape()
[4,3,1]
sage: w.RS_partition()
[4,3,1]
sage: w.longest_increasing_subsequences()
sage: pp.maximal_chains()
```

PROBLEM I (part 2)

Is there a "localized" version of the Greene-Kleitman invariant?

Can we find an interpretation of $A_k^L$ and $D_k^L$ in a poset?

Start by observing patterns of $A_k^L, D_k^L$ for the poset $I(\pi)$.

Note: $D_1^L = D_1$, but in general $D_k^L \neq D_k$ and $A_k^L \neq A_k$.

Possible initial tasks

- Choose a few permutations $\pi$ whose $\lambda^L$ and $\mu^L$ we have computed before. Try to choose permutations with $\lambda_2^L > 1$.
- Draw the Hasse diagram of $I(\pi)$ using the $n \times n$ square and "shadow" def.
- Study the set of ascents for $\pi$. Describe it as a subset of the poset $I(\pi)$.

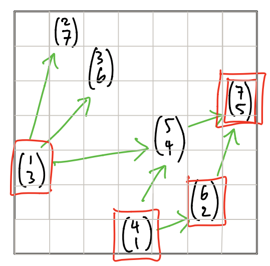
## Example

$\pi = \begin{pmatrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 3 & 7 & 6 & 1 & 4 & 2 & 5 \end{pmatrix}$. Compute the localized versions $\lambda^L$ and $\mu^L$.

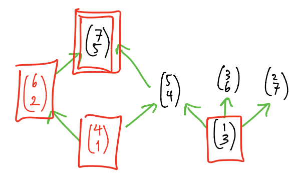
$$A_1^L = \operatorname{asc}(\pi) + 1$$

$$= \left| \left\{ \boxed{\tbinom{1}{3}}\ \cancel{\tbinom{2}{7}}\ \cancel{\tbinom{3}{6}}\ \boxed{\tbinom{4}{1}}\ \cancel{\tbinom{5}{4}}\ \boxed{\tbinom{6}{2}}\ \boxed{\tbinom{7}{5}} \right\} \right| = 4$$

Always unique, by def of asc(π)



π= 3 7 6 1 4 2 5



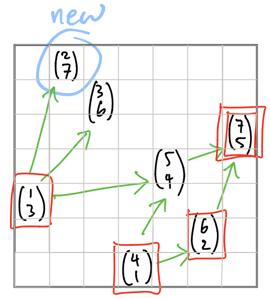
$$A_2^L = \max_{u_1, u_2} \left( \operatorname{asc}^*(u_1) + \operatorname{asc}^*(u_2) \right)$$

over all disjoint subsequences of π

$$= \left| \left\{ \boxed{\tbinom{1}{3}\ \tbinom{2}{7}}\ \cancel{\tbinom{3}{6}}\ \cancel{\tbinom{4}{1}}\ \cancel{\tbinom{5}{4}}\ \cancel{\tbinom{6}{2}}\ \cancel{\tbinom{7}{5}}, \quad \cancel{\tbinom{1}{3}}\ \cancel{\tbinom{2}{7}}\ \cancel{\tbinom{3}{6}}\ \tbinom{4}{1}\ \cancel{\tbinom{5}{4}}\ \tbinom{6}{2}\ \tbinom{7}{5} \right\} \right|$$

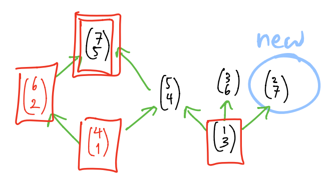
$$= 5$$

new



π= 3 7 6 1 4 2 5

new



$$A_3^L = \max_{u_1, u_2, u_3} \left( \operatorname{asc}^*(u_1) + \operatorname{asc}^*(u_2) + \operatorname{asc}^*(u_3) \right) = 6$$

disjoint subsequences of π

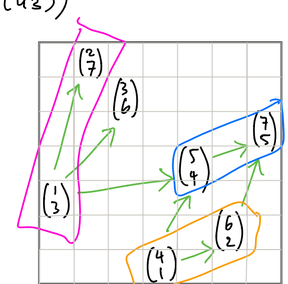
We can do, for example

in one subsequence

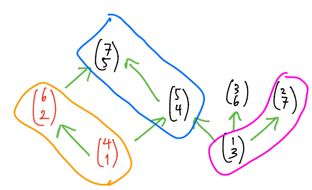
$$\boxed{\tbinom{1}{3}\ \tbinom{2}{7}}\ \cancel{\tbinom{3}{6}}\ \tbinom{4}{1}\ \tbinom{5}{4}\ \tbinom{6}{2}\ \tbinom{7}{5}$$

one subsequence

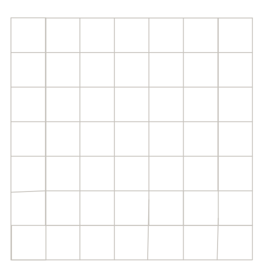
in one subsequence



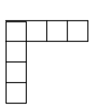
π= 3 7 6 1 4 2 5



$A_4^L = 7$



$\lambda^L = 4, 1, 1, 1$



# Update to PROBLEM I (part 2) (Wed, June 3)

I think other questions are more promising for this summer.

Reasoning is described below.

Perhaps the correct thing to do is to associate the BBS soliton partition to a naturally-labeled poset (that is, a poset together with a linear extension), but I have no specific conjectures.

Recall the map pp from $S_n$ to posets by assigning $\pi \in S_n$ to its inversion poset of permutation poset.

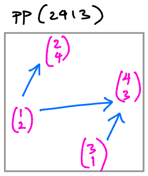
```
sage: Permutation([2,4,1,3]).permutation_poset()
```

Note that this map pp is not injective.

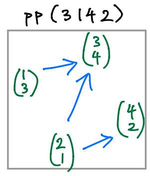
For example,

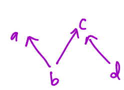
```
sage: Permutation([2,4,1,3]).permutation_poset().
      is_isomorphic(Permutation([3,1,4,2]).permutation_poset())
True
sage: view(Permutation([2,4,1,3]).permutation_poset())
sage: view(Permutation([3,1,4,2]).permutation_poset())
```

pp(2413)




pp(3142)




Furthermore, the poset 

shows that the BBS partition is not an invariant of a permutation poset / inversion poset of a permutation.

↳ An invariant of a math object is a value that does not change.

The BBS partition of $\pi = 2413$ is $\mu^L = (2,1,1)$, BBS soliton: $2 \ldots 1 \ldots 43$ at $t=1$.

—— " —— of $\pi = 3142$ is $\mu^L = (2,2)$, BBS soliton: $3142$ at $t=0$.