PROBLEM 1 Notes, part 1

Ref:
- Stanley "Enumerative Combinatorics Vol 1" 2nd ed
- Sage Math documentation (Google "Sage Math" + a math concept)
- Blog post "Localized version of Greene's thm" by Joel B. Lewis
- Lewis, Lyu, Pylyavskyy, Sen "Scaling limit of soliton lengths in a multicolor box-ball system" preprint 2019

Links on egunawan.github.io/REU/literature

(Last updated: 5 pm Thurs, June 11, 2020)

"Sage Practice" or "Practice" is a routine exercise you should do on your own, to help you become familiar with the problem.

"REU Exercise" is a more challenging problem which may help you eventually solve the PROBLEM.

- You should save all your "REU exercises" because they are likely to go into future presentations / reports / papers.

- During the Combinatorics group meeting, you will volunteer to present your answers or progress to the "REU exercises"

1.1 The Symmetric group

Def Write $[n] := \{1, 2, \ldots, n\}$ for $n \geq 1$.

A permutation is a bijection from $[n]$ to itself.

- Two-line notation $\pi = \begin{matrix} 1 & 2 & \cdots & n \\ \pi(1) & \pi(2) & \cdots & \pi(n) \end{matrix}$, e.g. $\pi = \begin{matrix} 1 & 2 & 3 & 4 & 5 & 6 & 7 \\ 4 & 2 & 3 & 6 & 5 & 1 & 7 \end{matrix}$

- One-line notation $\pi = \pi(1)\ \pi(2)\ \cdots\ \pi(n)$, e.g $\pi = 4\ 2\ 3\ 6\ 5\ 1\ 7$

- Cycle notation e.g $\pi = (146)(2)(3)(5)(7)$
  $= (146)$

Def The symmetric group S_n consists of permutations of $[n]$ using compositions as group operation.

- This summer, let's agree to multiply permutations from right to left (like function compositions).

  e.g. $(146)(14) = (16)$

  $(14)(146) = (46)$

Sage Practice : Do both computation in Sage. Follow Judson's examples.

```
sage: x = Permutation('(1,4)(2,5,6)')
sage: y = Permutation([4,5,3,1,6,2])
sage: x == y
True
sage: z = Permutation((4,5,3,1,6,2))
sage: w = Permutation("(4,5,3,1,6,2)")
sage: z == w
True
sage: x * w    # use * to do multiplication of permutations.
```

You can go from one-line notation to cycle notation.

```
sage: pi = Permutation([4,2,3,6,5,1,7])
sage: pi.cycle_string()
sage: pi.to_cycles()
```

Sage Practice : Does Sage multiply permutations from right to left, or from left to right?

1.2. Partitions

Def • A *partition* of $n \in \mathbb{Z}_{\geq 0}$ is a sequence $\lambda = (\lambda_1, \lambda_2, \dots)$ of integers λ_i where the λ_i are weakly decreasing, that is,

$$\lambda_1 \geqslant \lambda_2 \geqslant \lambda_3 \geqslant \dots,$$

and $\lambda_1 + \lambda_2 + \dots = n$.

- If $\lambda_k > 0$, and $\lambda_{k+1} = \lambda_{k+2} = \dots = 0$, write $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_k)$.

  The nonzero terms $\lambda_1, \lambda_2, \dots, \lambda_k$ are called *parts*. Say λ has k parts. The number of parts is called the *length* of λ, denoted $\ell(\lambda)$.

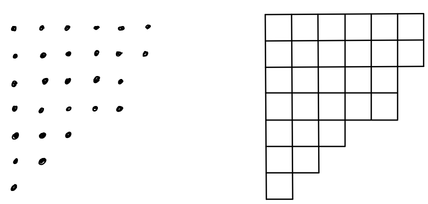
- If λ is a partition of n, write $\lambda \vdash n$ or $|\lambda| = n$.

Def • The *Ferrers diagram of λ* is an array of dots having k left-justified rows with row i containing λ_i dots for $1 \leq i \leq k$.

- The *Young diagram of λ* is the Ferrers diagram with dots replaced by boxes.
- We also write the *shape of λ*, $sh(\lambda)$, to describe the above.

Note: This is the "English" convention. There are also "French" and "Russian" notations.

E.g



Ferrers diagram and Young diagram of the partition $\lambda = (6, 6, 5, 5, 3, 2, 1)$.

Def The *conjugate (or transpose) partition* of λ is the partition λ^c whose shape is obtained from $sh(\lambda)$ by interchanging rows and columns. Equivalently, the shape of λ^c is the reflection of $sh(\lambda)$ about the main diagonal.

Rem λ_i^c = # of parts of λ of size larger than or equal to i.
λ_i = # of parts of λ^c of size larger than or equal to i.

Prop Suppose $\lambda = (\lambda_1, \lambda_2, \dots, \lambda_r)$ and $\mu = (\mu_1, \mu_2, \dots)$ are partitions of n such that the number of parts of μ that equal i is $\lambda_i - \lambda_{i+1}$. Then $\lambda^c = \mu$.

E.g. If $\lambda = (4, 3, 1, 1, 1)$, then $\lambda^c = (5, 2, 2, 1)$.

Sage Exercise: Compute examples with Sage

```
sage: lamb = Partition([4,3,1,1,1])
sage: print(lamb.ferrers_diagram())

sage: lamb.conjugate()
[5,2,2,1]
sage: lamb.conjugate() == Partition([5,2,2,1])
True
```

1.3 Schensted's theorem

Definitions

- For any $k \geq 0$, let A_k be the maximum size of the disjoint union of k increasing subsequences of w.

  E.g. $\pi = \begin{matrix} 1 & 2 & 3 & 4 & 5 \\ 3 & 4 & 5 & 1 & 2 \end{matrix}$ (two-row notation) $\pi = 34512$ (one-line notation)

$A_0 = 0$, $A_1 = 3$ (witnessed by 345), $A_2 = 5$ (witnessed by 345, 12), $A_k = 5$ $\forall\ k \geq 2$

- For $k \geq 1$, let $\lambda_k := A_k - A_{k-1}$.
- Let $\lambda = (\lambda_1, \lambda_2, \ldots)$
- Sage Exercise: We can compute λ in Sage (Try it).

```
Sage: Permutation([3,4,5,1,2]).RS_partition()
[3,2]
Sage: print(_.ferrers_diagram())
• • •
• •
```

- For any $k \geq 0$, let D_k be the maximum size of the disjoint union of k decreasing subsequences.
- For $k \geq 1$, let $\mu_k := D_k - D_{k-1}$
- Let $\mu = (\mu_1, \mu_2, \mu_3, \ldots)$

  E.g for the same $\pi = 34512$

$D_0 = 0$, $D_1 = 2$ (witnessed by 31), $D_2 = 4$ (witnessed by 31, 42), $D_k = 5$ $\forall\ k \geq 3$

$\mu = (2, 2, 1)$

• •
• •
•

Note $\lambda = (3,2)$ and $\mu = (2,2,1)$ are conjugate partitions.

Practice Let $\pi = 7\,9\,5\,8\,4\,6\,3\,2\,1$. Compute λ and μ by hand.

Use Sage to verify your answers.

```
sage: Permutation([7,9,5,8,4,6,3,2,1]).RS_partition()
sage: Permutation([7,9,5,8,4,6,3,2,1]).RS_partition().conjugate()
```

λ μ

Thm (Schensted 1961 "Longest increasing and decreasing subsequences")

Let π be a permutation in S_n. Then the length of the longest increasing subsequence of π is equal to the size of the first row of the RS_partition of π (I have not defined yet), and the length of the longest decreasing subsequence of π is equal to the size of the first column of the RS_partition of π.

Thm (Greene 1974 "An extension of Schensted's theorem")

Let π be a permutation in S_n. Let λ and μ be as defined above.

Then:

(i) λ is a partition of n, i.e. λ is a weakly decreasing sequence of nonnegative integers with sum n.

(ii) μ is a partition of n

(iii) λ and μ are conjugate partitions, that is,
μ_i = # of parts of λ of size larger than or equal to i,
λ_i = # of parts of μ of size larger than or equal to i.

Call λ the Robinson-Schensted (RS) partition

Why is this partition interesting? Connection to representation theory of the symmetric group, and other mathematics.

1.4 Localized version of the Robinson – Schensted (RS) partition

Ref: Lewis' 2019 OPAC blog

1.4.1 Localized version of union of increasing subsequences

<u>Def</u> • An <u>ascent</u> in a sequence $u = (u_1, u_2, \ldots)$ is an index i s.t $u_i < u_{i+1}$.

• Let $asc(u)$ denote the number of ascents of u

• Let $asc^*(u) = 1 + asc(u)$ if u is nonempty.
$asc^*(u) = 0$ if u is the empty sequence.

e.g. If $u = (3, 4, 5, 1, 2)$, the ascents of u are the indices 1, 2, 4.
So $asc(u) = 3$.
$asc^*(u) = 4$.

<u>Def</u> If $\pi \in S_n$, define

• $$A_k^L = \max_{u_1, \ldots, u_k} \left(asc^*(u_1) + \cdots + asc^*(u_k) \right)$$

where the maximum is taken over disjoint subsequences of π

• $\lambda^L = (\lambda_1^L, \lambda_2^L, \ldots)$ where $\lambda_k^L := A_k^L - A_{k-1}^L$ for $k \geqslant 1$.

E.g. $\pi = 3\ 4\ 5\ 1\ 2$ (from before)

$A_0^L = 0$, $A_1^L = asc^*(\pi) = 4$, $A_2^L = asc^*(345) + asc^*(12) = 3 + 2 = 5$, $A_k^L = 5$ $\forall$ $k \geqslant 2$.

A subsequence does <u>not</u> need to be consecutive.

Here $\lambda^L = (4, 1)$

1.4.2 Localized version of union of decreasing subsequences

Def • If $u = (u_1, u_2, \dots)$ is a sequence, let

$d(u)$ = the maximum size of a decreasing subsequence of u.

E.g. If $u = (3, 4, 5, 1, 2)$, then a longest decreasing subsequence is $(3, 1)$
So $d(u) = 2$

Def If $\pi \in S_n$, define

- $$D_k^L = \max_{\pi = u_1 | u_2 | \cdots | u_k} \left(d(u_1) + d(u_2) + \cdots + d(u_k) \right)$$

  where the maximum is taken over ways of writing π as a concatenation $u_1 | u_2 | \cdots | u_k$ of consecutive subsequences.

- $M^L = (M_1^L, M_2^L, \dots)$ where $M_k^L := D_k^L - D_{k-1}^L$ for $k \geq 1$.

E.g. $\pi = 3\ 4\ 5\ 1\ 2$

$D_0^L = 0$, $D_1^L = d(34512) = 2$,

$$D_2^L = d(3) + d(4512) = 1 + 2 = 3$$ other possible divisions are $34|512$ and $3451|2$

$$D_3^L = d(3) + d(4) + d(512) = 1 + 1 + 2 = 4$$ other possible divisions are $3|451|2$, $34|51|2$

$$D_4^L = d(3) + d(4) + d(51) + d(2) = 1 + 1 + 2 + 1 = 5$$

$D_k^L = 5$ for all $k \geq 4$.

So $M^L = (2, 1, 1, 1)$.

Note: $\lambda^L = (4, 1)$ and $M^L = (2, 1, 1, 1)$ are conjugate partitions.

1.4.3 Localized version of Greene's Thm

Theorem (Lemma 2.1 of Lewis, Lyu, Pylyavskyy, Sen [LLPS] 2019 "Scaling limit of soliton lengths in a multicolor box-ball system")

Let π be a permutation in S_n. Let λ^L and μ^L be as defined above.

Then:

(i) λ^L is a partition of n, i.e. λ is a weakly decreasing sequence of nonnegative integers with sum n.

(ii) μ^L is a partition of n

(iii) λ^L and μ^L are conjugate partitions, that is,
μ_i^L = # of parts of λ of size larger than or equal to i,
λ_i = # of parts of μ of size larger than or equal to i.

- Call μ^L the box-ball system (BBS) soliton partition of π.

Optional task

- Write a python function which computes the A_K^L's, D_K^L's, λ^L, μ^L ← the BBS soliton partition
- The function should interact with Sage. It should take the Permutation object as input and output the Partition object
- put your functions in a .py file (if you're using the Terminal) or in a cell (if you use a worksheet / CoCalc)
- If you use the Terminal, you can call the file from Sage by typing: attach('~/path to file / yourfile.py')
- Share the file with everyone.

- Why is this theorem interesting? The partition μ^L describes the long-term behavior of a multicolor box-ball system initialized with one ball in each color $\{1, 2, \dots, n\}$, arranged according to π.

The BBS is a dynamical system consisting of balls in an infinite strip.

↳ Balls take turns jumping to the first available cell, in decreasing order starting with the largest-numbered ball. Use 0's to denote the empty cells.

Eg. $\pi = 3\ 4\ 5\ 1\ 2$

$t = 0$ 3 4 5 1 2 0 0 0 ... → initial configuration
3 4 0 1 2 5 0 0 ...
3 0 4 1 2 5 0 0 ...
0 3 4 1 2 5 0 0 ...
0 3 4 1   5 2 0 ...
$t = 1$ 0 3 4 0 1 5 2 0 ... → configuration after one BBS move
0 3 4 0 1 0 2 5 0 ..
0 3 0 4 1 0 2 5 0 ...
0 0 3 4 1 0 2 5 0 ...
0 0 3 4 1 0 0 5 2 0 ...
$t = 2$ 0 0 3 4 0 1 0 5 2 0 ... → configuration after two BBS moves
0 0 3 4 0 1 0 0 2 5 0 ...
0 0 3 0 4 1 0 0 2 5 0 ...
0 0 0 3 4 1 0 0 2 5 0 ...
0 0 0 3 4 1 0 0 0 5 2 0 ...
$t = 3$ 0 0 0 3 4 0 1 0 0 5 2 0 ... → configuration after three BBS moves
⋮

Observe

$t = 0$ 3 4 5 1 2 0 0 ...
→ $t = 1$ 0 3 4 0 1 5 2 0 ..
$t = 2$ 0 0 3 4 0 1 0 5 2 0 ..
$t = 3$ 0 0 0 3 4 0 1 0 0 5 2 0 ..
$t = 4$ 0 — — 0 3 4 0 1 0 0 0 5 2 0 ...
$t = 5$ 0 - - 0 3 4 0 1 0 0 0 0 5 2 0 ..
$t = 6$ 0 - - 0 3 4 0 1 0 0 0 0 0 5 2 0 ..
$t = 7$ 0 - - - 0 3 4 0 1 0 0 0 0 0 0 5 2 0 ..
$t = 8$ 0 — - - 0 3 4 0 1 0 0 0 0 0 0 0 0 5 2 0 ...
⋮

At every time-step starting at $t = 1$,
the two balls 5 2 advances two jumps to the right,
the ball 1 advances one jump to the right,
the ball 4 advances one jump to the right,
the ball 3 advances one jump to the right

} giving $M^L = (2,1,1,1)$
5 2
1
4
3

Optional task: Figure out if this is implemented in Sage

E.g. $\pi = 6\ 3\ 4\ 1\ 7\ 2\ 8\ 5$

$t = 0$ 6 3 4 1 7 2 8 5 0 0 0 ...
→ $t = 1$ 0 0 3 0 6 1 7 4 8 5 2 0 ..
$t = 2$ 0 0 0 3 0 0 6 1 7 4 0 8 5 2 0 ...
$t = 3$ 0 0 0 0 3 0 0 0 6 1 7 4 0 0 8 5 2 0 ...
⋮

At every time-step starting at $t = 1$,
the three balls 8 5 2 advances three jumps to the right
the two balls 7 4 — " — two jumps — " —
the two balls 6 1 — " — two jumps — " —
the one ball 3 — " — one jump — " —

giving $\mu^L = (3,2,2,1)$

8 5 2
7 4
6 1
3

Practice: • Check that $\mu^L = (3,2,2,1)$ for $\pi = 6\ 3\ 4\ 1\ 7\ 2\ 8\ 5$, using the definition of D^L_K.

• Compute λ^L for $\pi = 6\ 3\ 4\ 1\ 7\ 2\ 8\ 5$, using the def of A^L_K. Confirm that your answer is the conjugate of μ^L.

Note: $\mu = \mu^L = (3,2,2,1)$ for $\pi = 6\ 3\ 4\ 1\ 7\ 2\ 8\ 5$, but this is not true in general, as you see with $\pi = 3\ 4\ 5\ 1\ 2$.

More examples

$\pi = 1\ 2\ 3\ 4\ 5$

$t=0$ 1 2 3 4 5 $M^L = (1,1,1,1)$

$t=1$ 1 2 3 4 5

5
4
3
2
1

$\pi = 4\ 3\ 2\ 1$

$t=0$ 4 3 2 1 $M^L = (4)$

$t=1$ 4 3 2 1

$\pi = 6\ 5\ 7\ 4\ 3\ 2\ 1$

$t=0$ 6 5 7 4 3 2 1

$t=1$ 6 7 5 4 3 2 1

$t=2$ 6 7 5 4 3 2 1

$M^L = (6,1)$

7 5 4 3 2 1
6

$\pi = 5\ 3\ 2\ 1 \mid 8\ 6 \mid 7\ 4$

$t=0$ 5 3 2 1 8 6 7 4

$t=1$ 5 3 6 2 8 7 4 1

$t=2$ 5 6 3 2 8 7 4 1

$t=3$ 5 6 3 2 8 7 4 1

$M^L = (4,3,1)$

8 7 4 1
6 3 2
5

$\pi = 5\ 3\ 2\ 1 \mid 8\ 7\ 4 \mid 6$

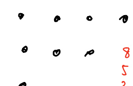
$t=1$ 5 3 2 4 8 7 6 1

$t=2$ 3 5 4 2 8 7 6 1

$t=3$ 3 5 4 2 8 7 6 1

$t=4$ 3 5 4 2 8 7 6 1

$M^L = (4,3,1)$



8 7 6 1
5 4 2
3

$\pi = 5\ 2\ 1 \mid 3 \mid 8\ 4 \mid 6 \mid 7$

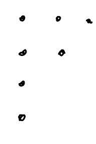
$t=1$ 2 5 3 4 6 8 7 1

$t=2$ 2 3 5 6 4 8 7 1

$t=3$ 2 3 5 6 4 8 7 1

$t=4$ 2 3 5 6 4 8 7 1

$M^L = (3,2,1,1,1)$



8 7 1
6 4
5
3
2

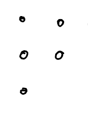
π 3 2 | 5 1 | 8 4 | 7 6

$t=1$ 3 5 2 4 1 8 7 6

$t=2$ 3 2 5 4 1 8 7 6

$t=3$ 3 2 5 4 1 8 7 6

$M^L = (3,3,1,1)$



8 7 6
5 4 1
2
3

π 3 1 | 5 2 | 8 4 | 7 6

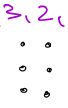
$t=1$ 3 1 5 2 4 8 7 6

$t=2$ 3 1 2 5 4 8 7 6

$t=3$ 1 3 2 5 4 8 7 6

$t=4$ 1 3 2 5 4 8 7 6

$M^L = (3,2,2,1)$



8 7 6
5 4
3 2
1

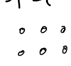
π 3 2 | 5 4 1 | 8 7 6

$t=0$ 3 2 5 4 1 8 7 6

$t=1$ 3 2 5 4 1 8 7 6

$t=2$ 3 2 5 4 1 8 7 6

$M^L = (3,3,2)$



8 7 6
5 4 1
3 2

Eg $\pi = 5\ 4\ 2\ 3\ 1$

$t=1$: 2 5 4 3 1

$t=2$: 2 5 4 3 1

$\pi = 5\ 2\ 4\ 3\ 1$

$t=1$: 2 5 4 3 1

$\pi = 3\ 2\ 5\ 4\ 1\ 7\ 6$

$t=1$: 3 2 5 4 7 6 1

$t=2$: 3 2 5 4 7 6 1

$\pi = 3\ 1\ 5\ 2 \mid 8\ 4 \mid 7\ 6$

$\pi = 7\ 6\ 8\ 4\ 5\ 2\ 3\ 1$

$t=1$: 7 4 2 8 6 5 3 1

$t=2$: 7 4 2 8 6 5 3 1

$\pi = 3\ 1\ 5\ 2\ 7\ 6\ 8\ 4$

$t=1$: 3 1 5 2 7 8 6 4

$t=2$: 3 1 5 7 2 8 6 4

$\boxed{t=3}$: 3 5 1 7 2 8 6 4

$t=4$: 3 5 1 7 2 8 6 4

$t=5$: 3 5 1 7 2 8 6 4

$\pi = 3\ 1\ 5\ 2 \mid 8\ 4 \mid 7\ 6$ (already did this example earlier)

$t=1$: 3 1 5 2 4 8 7 6

$t=2$: 3 1 2 5 4 8 7 6

$t=3$: 1 3 2 5 4 8 7 6

$t=4$: 1 3 2 5 4 8 7 6

E.g.

$\pi = 1\ 7\ 2\ 3\ 5\ 9\ 4\ 6\ 8$

- $A_k^L = \max\limits_{u_1,\dots,u_k} \left(asc^*(u_1) + \dots + asc^*(u_k)\right)$

  where the maximum is taken over disjoint subsequences of π
- $\lambda^L = (\lambda_1^L, \lambda_2^L, \dots)$ where $\lambda_k^L := A_k^L - A_{k-1}^L$ for $k \geq 1$.

$$A_1^L = asc(1\ 7\ 2\ 3\ 5\ 9\ 4\ 6\ 8) + 1$$
$$= 6 + 1 = 7$$

$$A_2^L = asc^*(1\ 7\ 2\ 3\ 5\ 9\ 4\ 6\ 8) + asc^*(1\ 7\ 2\ 3\ 5\ 9\ 4\ 6\ 8)$$
$$= (4+1) + (2+1) = 8$$

$$A_3^L = asc^*(1\ 7\ 2\ 3\ 5\ 9\ 4\ 6\ 8) + asc^*(1\ 7\ 2\ 3\ 5\ 9\ 4\ 6\ 8) + asc^*(1\ 7\ 2\ 3\ 5\ 9\ 4\ 6\ 8)$$
$$= (4+1) + (2+1) + 1 = 9$$

Some other ways to break π into three increasing subsequences include:

1 7 2 3 5 9 4 6 8
1 7 2 3 5 9 4 6 8
1 7 2 3 5 9 4 6 8

1 7 2 3 5 9 4 6 8
1 7 2 3 5 9 4 6 8
1 7 2 3 5 9 4 6 8

1 7 2 3 5 9 4 6 8
1 7 2 3 5 9 4 6 8
1 7 2 3 5 9 4 6 8

1 7 2 3 5 9 4 6 8
1 7 2 3 5 9 4 6 8
1 7 2 3 5 9 4 6 8

$\lambda^L = \dots\dots$

$\vdots$

- $D_k^L = \max\limits_{\pi = u_1|u_2|\cdots|u_k} \left(d(u_1) + d(u_2) + \dots + d(u_k)\right)$

  where the maximum is taken over ways of writing π as a concatenation $u_1|u_2|\cdots|u_k$ of consecutive subsequences.
- $M^L = (M_1^L, M_2^L, \dots)$ where $M_k^L := D_k^L - D_{k-1}^L$ for $k \geq 1$.

$$D_1^L = d(1\ 7\ 2\ 3\ 5\ 9\ 4\ 6\ 8)$$
$$= 3$$

unique longest dec. subseq

$$D_2^L = d(1 | \cancel{7\ 2\ 3\ 5\ 9\ 4\ 6\ 8}) \quad d(\cancel{1} | 7\ 2\ 3\ 5\ 9\ 4\ 6\ 8)$$
$$= 1 + 3$$

$$D_3^L = d(1 | \cancel{7\ 2\ 3\ 5\ 9\ 4\ 6} | \cancel{8}) + d(\cancel{1} | 7\ 2\ 3\ 5\ 9\ 4\ 6 | \cancel{8}) + d(\cancel{1} | \cancel{7\ 2\ 3\ 5\ 9\ 4\ 6} | 8)$$
$$= 1 + 3 + 1$$

$\vdots$

$$D_7^L = d(1 | 7\ 2 | 3 | 5 | 9\ 4 | 6 | 8)$$
$\vdots$
$$= 1 + 2 + 1 + 1 + 2 + 1 + 1 = 9$$

Check w/ BBS moves

$t=0$	1	7	2	3	5	9	4	6	8	∘	∘	∘	∘	∘	∘	∘	
$t=1$	∘	1	∘	2	3	7	∘	5	6	9	8	4	∘	∘	∘	∘	
$t=2$	∘	∘	1	∘	2	3	7	∘	5	6	∘	∘	9	8	4	∘	
$t=3$	∘	∘	∘	1	∘	2	3	7	∘	5	6	∘	∘	∘	∘	9	8	4

$M^L = (3, 1, 1, 1, 1, 1, 1)$

$\dots$

$\vdots$

E.g.

E.g. $\pi = 7\ 9\ 5\ 8\ 4\ 6\ 3\ 2\ 1$

- $A_k^L = \max_{u_1,\dots,u_k} (asc^*(u_1) + \dots + asc^*(u_k))$
where the maximum is taken over disjoint subsequences of π
- $\lambda^L = (\lambda_1^L, \lambda_2^L, \dots)$ where $\lambda_k^L := A_k^L - A_{k-1}^L$ for $k \geqslant 1$.

$$A_1^L = asc(\pi) + 1 = \#\{1,3,5\} + 1 = 3 + 1 = 4$$

$$A_2^L = asc^*(7\,9\,5\,8\,4\,6\,3\,2\,1) + asc^*(7\,9\,5\,8\,4\,6\,3\,2\,1) = (3+1) + (0+1) = 5$$

$$A_3^L = asc^*(7\,9\,5\,8\,4\,6\,3\,2\,1) + asc^*(7\,9\,5\,8\,4\,6\,3\,2\,1) + asc^*(7\,9\,5\,8\,4\,6\,3\,2\,1) = (3+1) + (0+1) + (0+1) = 6$$

$$A_4^L = asc^*(7\,9\,5\,8\,4\,6\,3\,2\,1) + asc^*(7\,9\,5\,8\,4\,6\,3\,2\,1) + asc^*(7\,9\,5\,8\,4\,6\,3\,2\,1) + asc^*(7\,9\,5\,8\,4\,6\,3\,2\,1) = (3+1) + (0+1) + (0+1) + (0+1) = 7$$

$$A_5^L = asc^*(7\,9\,5\,8\,4\,6\,3\,2\,1) + asc^*(7\,9\,5\,8\,4\,6\,3\,2\,1) + asc^*(7\,9\,5\,8\,4\,6\,3\,2\,1) + asc^*(7\,9\,5\,8\,4\,6\,3\,2\,1) + asc^*(7\,9\,5\,8\,4\,6\,3\,2\,1) = (2+1) + (1+1) + 1 + 1 + 1 = 8$$

$$A_6^L = asc^*(7\,9\,5\,8\,4\,6\,3\,2\,1) + asc^*(7\,9\,5\,8\,4\,6\,3\,2\,1) + asc^*(7\,9\,5\,8\,4\,6\,3\,2\,1) + asc^*(7\,9\,5\,8\,4\,6\,3\,2\,1) + asc^*(7\,9\,5\,8\,4\,6\,3\,2\,1) + asc^*(7\,9\,5\,8\,4\,6\,3\,2\,1) = (1+1) + (1+1) + (1+1) + 1 + 1 + 1 = 9$$

$$\lambda^L = (4,1,1,1,1,1)$$

- $D_k^L = \max_{\pi = u_1|u_2|\cdots|u_k} (d(u_1) + d(u_2) + \dots + d(u_k))$
where the maximum is taken over ways of writing π as a concatenation $u_1|u_2|\cdots|u_k$ of consecutive subsequences.
- $M^L = (M_1^L, M_2^L, \dots)$ where $M_k^L := D_k^L - D_{k-1}^L$ for $k \geqslant 1$.

D_1^L = maximum size of decreasing subsequence of π

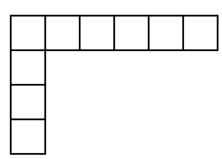
$$= \#\{7\,9\,5\,8\,4\,6\,3\,2\,1\} = 6$$

$$D_2^L = d(7\,9\,5|8\,4\,6\,3\,2\,1) + d(7\,9\,5|8\,4\,6\,3\,2\,1) = 2 + 5 = 7$$

$$D_3^L = d(7|9\,5|8\,4\,6\,3\,2\,1) + d(7|9\,5|8\,4\,6\,3\,2\,1) + d(7|9\,5|8\,4\,6\,3\,2\,1) = 1 + 2 + 5 = 8$$

$$D_4^L = d(7|9\,5|8\,4|6\,3\,2\,1) + d(7|9\,5|8\,4|6\,3\,2\,1) + d(7|9\,5|8\,4|6\,3\,2\,1) + d(7|9\,5|8\,4|6\,3\,2\,1) = 1 + 2 + 2 + 4 = 9$$

$$M^L = (6,1,1,1)$$



Verify M^L using Box-ball system

t=0 7 9 5 8 4 6 3 2 1
t=1 7 5 4 9 8 6 3 2 1
t=2 7 5 4 9 8 6 3 2 1

REU PROBLEM I

Prove that λ^L and μ^L are partitions, and prove that they are conjugate.
Give a proof that stays in the realm of permutation combinatorics
(i.e. not using box-ball system theory)

REU Exercise 1 (without using [LLPS] Lemma 2.1)

i. Prove that λ_1^L equals the number of positive parts of μ^L. ← Look at the sketch of proof & add/edit

ii. Prove that μ_1^L equals the number of positive parts of λ^L.

(# of parts of λ^L) $\geq \mu_1^L$ is done using pigeonhole principle, see Group Meeting Thurs May 28

REU Exercise 2 (possibly problem) — Use Fukuda Convention

(i) Given $\pi \in S_n$, at what time t do the balls get sorted into their long-term behavior?

data using LLPS convention

For example,

$t = 0$ for $\pi = 5\,4\,3\,2\,1$, $\pi = 1234$, $\pi = 3\,2\,5\,4\,1\;\;8\,7\,6$

$t = 1$ for $\pi = 6\,5\,7\,4\,3\,2\,1$, $\pi = 3\,4\,5\,1\,2$, $\pi = 6\,3\,4\,1\;\;7\,2\,8\,5$

$t = 2$ for $\pi = 5\,2\,1\,3\,8\,4\,6\,7$

$t = 3$ for $\pi = 3\,1\,5\,2\,8\,4\,7\,6$

(End of lecture on Tues, May 26, 2020)

(ii) What permutations guarantee $t = 0$?

(iii) What is an upper bound of t?

(iv) Which permutations have largest t?

(v) (Asymptotic question) How long does a typical permutation take?

(vi) Given a BBS system which contains $\pi \in S_n$ as one of the states, how many permutations are part of the system?

Notes for (vi)

$t=-2$

$t=-1$

$t=0$ $\qquad$ 75862413

number of permutation configurations in this system is 3

Notes for (iii)

up to $n=9$,

n is an upper bound

$n-3$ is the minimum upper bound (checked for $4 \le n \le 9$)

n	min upp bound of t
1	0
2	0
3	1
4	1
5	2
6	3
7	4
8	5
9	6

$\boxed{\text{done}}$ REU Exercise 3

Given $\pi \in S_n$, let reverse(π) be the permutation $\pi_n \pi_{n-1} \cdots \pi_2 \pi_1$.

Is $\mu^L(\pi) = \lambda^L(\text{reverse}(\pi))$? Ans No. e.g $n=4$, 2143, 3142, 2413, 3412.

$\lambda = \lambda^L = (2,2)$

$\lambda = (2,2)$

$\lambda^L = (3,1)$

* Practice: Convince yourself that $\mu(\pi) = \lambda(\text{reverse}(\pi))$. (Classical version)

1.5 Partial order (poset)

Def If P is a set, then a partial order on P is a relation $\leqslant$ s.t.

- $a \leq a$ (reflexive)
- $a \leq b$ and $b \leq a$ implies $a = b$ (antisymmetry)
- $a \leq b$ and $b \leq c$ implies $a \leq b$ (transitivity)

Say that $(P, \leq)$ is a partially ordered set (poset)

Def (Dominance partial order on partitions of n)

Suppose $p = (p_1, p_2, \dots)$ and $q = (q_1, q_2, \dots)$ are partitions of n.

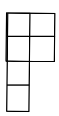
Then $p \leq q$,

say "q dominates p" iff

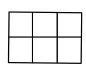
for all $k \geqslant 1$, $p_1 + p_2 + \dots + p_k \leq q_1 + q_2 + \dots + q_k$.

Intuitively, if $p \leq q$, $sh(p)$ is tall and skinny, $sh(q)$ is short and fat.

E.g. $(2,2,1,1) \leq (3,3)$



$2 \leq 3$
$2+2 \leq 3+3$
$2+2+1 \leq 3+3+0$
$2+2+1+1 \leq 3+3+0+0$



E.g. $(4,1,1)$ and $(3,3)$ are not comparable:

$4 \geqslant 3$
$4+1 \leqslant 3+3$

E.g. $(1,1,\dots,1) \leq q$ for all partitions q of n.

$p \leq (n)$ for all partitions p of n.

Sage practice

```
sage: p = Partition([2,2,1,1])
sage: q = Partition([3,3])
sage: q.dominates(p)
True
sage: q.dominated_partitions()
```

$* \lambda^L_1 \geq (\mu^L)^c_1$ (by REU Exercise 1(i), they are equal)

REU Exercise 4 $\lambda^L \geq (\mu^L)^c$ $*$Start with $k=2$

Show that λ^L dominates the conjugate of μ^L (without [LLPS] Lemma 2.1)

I.e. show that the conjugate of μ^L is smaller or equal to λ^L in the dominance order.

Note: If we show $\lambda^L \leq (\mu^L)^c$, we would solve PROBLEM 1. —end of Problem 1, part 1 notes—

Wed, May 27, 2020

To compute RSK algorithm :

```
sage : pi = Permutation([ ... ])
sage : RSK(pi)        # the shape of RSK(pi) = λ(pi)
sage : pi.RS_partition()
```

classical