Notes on "backward RSK" and "backward soliton decomposition

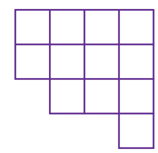
(Updated Mon, July 6, 2020)

In this document, we call it the "backward RS row insertion"

Def

In this document, we say that a "backward Young diagram" is the usual English notation Young diagram, but shifted so that all rows are right aligned

(4, 4, 3, 1)



A "backward tableau" is a filling of a "backward Young diagram" with $\{1, 2, \ldots, n\}$ so that the rows are columns are increasing.

Backward RS row insertion algorithm (for convention #1)

Let $w \in S_n$. Starting from w_n, then $w_{n-1}, \cdots, w_1$, insert each letter x in w into a partial "backward tableau" P using the following rule.

Let $R :=$ first row of P

WHILE x is bigger than some element of R, DO

$y :=$ the largest element of R smaller than x

Replace y by x (in the "backward" tableau P)

$x := y$

$R :=$ next row of P

END WHILE LOOP

Now, x is smaller than every element of R, so place x at the beginning of row R.

Note:
- To compute this the existing RSK code in Sage, read below.
- To get Convention #2 or #3, need to describe different algorithm

Example of "backward RS" map

$W = 5\ 9\ 4\ 8\ 7\ 3\ 6\ 1\ 2 \longrightarrow$

P	Q
2	9
1 2	8 9

1 6
  2
	8 9
	  7

3 6 1 2	8 9 6 7

3 7
1 6
  2
	8 9
	6 7
	  5

3 8
1 7
  6
  2
	8 9
	6 7
	  5
	  4

4 8
3 7
1 6
  2
	8 9
	6 7
	3 5
	  4

4 9
3 8
1 7
  6
  2
	8 9
	6 7
	3 5
	  4
	  2

Or maybe it's more natural to write

Conv #2

	2
1	6
3	7
4	8
5	9

or

possible conv #3

9	5
8	4
7	3
6	1
2	

= Complement

1	5
2	6
3	7
4	9
8	

= complement(ev(P))

Convention #1

Backward P

5	9
4	8
3	7
1	6
	2

Backward Q

8	9
6	7
3	5
1	4
	2

Here, the "backward row reading word" is 5 9 4 8 3 7 1 6 2 read from north west corner, $R_1 R_2, \ldots R_h$ where R_k is the k-th row.

Reason why I care.

- Fukuda uses the carrier algorithm and Knuth transformations to prove that the usual RSK insertion P-tableau is a conserved quantity of a Box-ball system.

  As a Corollary, the "backward P-tableau" is also a conserved quantity of a Box-ball system.

- Furthermore, we have the following ~~conjecture~~ fact (due to Dihedral symmetry):

  If the BBS decomposes into solitons of shape RS, then ⓐ the "forward" soliton decomposition is the RS P-tableau } Checked up to n = 11

  AND

  ⓑ the "backward" soliton decomposition is the "backward" P-tableau } Checked up to n=11 also

→ Part ⓐ is proved for BBS system which contains a (usual RS) row reading word.

Example 1

"Backward" soliton decomposition = "backward" P-tableau =

5	9
4	8
3	7
1	6
	2

possible convention #1

or better notation?

	2
1	6
3	7
4	8
5	9

possible convention #2 (Same as forward tableau)

= Rot 180° (Compl (ev P))

possible conv #3

9	5
8	4
7	3
6	1
2	

possible conv #3

t=-4	5	9	4	8	3	7	1	6	0	0	2														
t=-3			5	9	4	8	3	7	1	6	0	2													
t=-2					5	9	4	8	3	7	1	6	2												
t=-1							5	9	4	8	3	7	6	1	2										
t=0 w =									5	9	4	8	7	3	6	1	2								
t=1											5	9	8	4	7	3	6	1	2						
t=2													9	5	8	4	7	3	6	1	2				
t=3														9	0	5	8	4	7	3	6	1	2		
t=4															9	0	0	5	8	4	7	3	6	1	2

Forward soliton decomposition = "Forward" usual P-tableau =

1	2
3	6
4	7
5	8
9	

Note: It is a coincidence that I have a "backward row word" at t = -2 and a usual row word at t = 2.

The conjecture does not require a reading word to be one of the configurations in the system.

Example 2

$t=-1$ 1 3 4 0 2 0 0 0 0 0
$t=0$ W= 1 3 2 4 0 0 0
$t=1$ 0 0 0 0 0 3 0 1 2 4

1	3	4
		2

is the backward soliton decomposition

1	2	4
3		

is the forward soliton decomposition

W= 1 3 2 4 → "backward RS"

P	Q
4	4
2 4	3 4
3 4 / 2	3 4 / 2

1	3	4
		2

1	3	4
		2

Backward RS P-tableau

"Forward" usual RS:

P =

1	2	4
		3

Q =

1	2	4
	3	

Using SageMath RSK to get "backward RSK"

W= 5 9 4 8 7 3 6 1 2

① w_{comp} = 5 1 6 2 3 7 4 9 8   w.complement()

RSK(w_{comp})

P:

1	2	3	4	8
5	6	7	9	

Q:

1	3	5	6	8
2	4	7	9	

② Replace P, Q with complement (P, Q)

comp(P) =

9	8	7	6	2
5	4	3	1	

comp(Q) =

9	7	5	4	2
8	6	3	1	

③ Take conjugate of comp(P) and comp(Q)

9	5
8	4
7	3
6	1
2	

9	8
7	6
5	3
4	1
2	

④ Reverse each row in result of step 3 (so that the rows are increasing)

5	9
4	8
3	7
1	6
	2

Convention #1

8	9
6	7
3	5
1	4
	2

w.reverse().complement()

① $w^{rc} = w^{\text{reverse and compl}}$ = 8 9 4 7 3 2 6 1 5

RSK(w^{rc})

P:

1	5
2	6
3	7
4	9
8	

Q:

1	2
3	4
5	7
6	9
8	

② Replace P, Q with complement (P, Q)

comp(P) =

9	5
8	4
7	3
6	1
2	

comp(Q) =

9	8
7	6
5	3
4	1
2	

③ Reverse each row in comp(P), comp(Q) (so that the rows are increasing)

Convention #1

5	9
4	8
3	7
1	6
	2

8	9
6	7
3	5
1	4
	2

w.reverse().complement()

① $w^{rc} = w^{\text{reverse and compl}}$ = 8 9 4 7 3 2 6 1 5

RSK(w^{rc})

P:

1	5
2	6
3	7
4	9
8	

Q:

1	2
3	4
5	7
6	9
8	

② Replace P, Q with complement (P, Q)

comp(P) =

9	5
8	4
7	3
6	1
2	

comp(Q) =

9	8
7	6
5	3
4	1
2	

③ Rotate 180°

	2
1	6
3	7
4	8
5	9

convention #2 (our pick)

(Lecture on Thurs Jun 25)
Week 5

Problem I notes : Evacuation

Ref: Sagan Sec 3.9

Evacuation is an operation to turns a tableau into another tableau of the same shape. There are many equivalent algorithms to do evacuation. One algorithm is described in Sagan Sec 3.9 using slides (pg 121-122).

Here is another algorithm: **Def 1 of evacuation tableau of a tableau**

$P =$

1	2
3	6
4	7
5	8
9	

Rotate P by 180° : $P^{rot} =$

	9
8	5
7	4
6	3
2	1

```
sage: w=Permutation([...])
sage: P=RSK(w)[0]
sage: P.evacuation()
```

"Take the complement" Subtract all values from n+1 $P' =$

	1
2	5
3	6
4	7
8	9

Perform Jeu de Taquin (JdT), see Sec 3.7 Sagan (correct the typos before reading!) or wikipedia entry for "Jeu de Taquin"

1	
2	5
3	6
4	7
8	9

1	5
2	
3	6
4	7
8	9

1	5
2	6
3	
4	7
8	9

1	5
2	6
3	7
4	
8	9

The evacuation tableau of P, denoted ev P, is $JdT(P') =$

1	5
2	6
3	7
4	9
8	

Observe:

backward RSK

9	5
8	4
7	3
6	1
2	

= complement

1	5
2	6
3	7
4	9
8	

Dihedral group of the 8 symmetries of a square

(n×n permutation matrix, or the xy coordinates from ProbI Part 2 notes / Sec 3.6 Sagan)

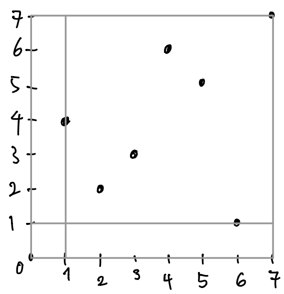
On this page, I use xy coordinates matrix

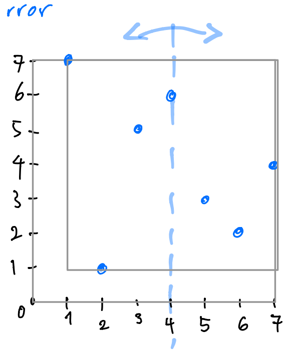
f_1

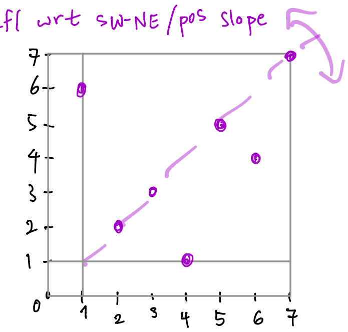
Reflection wrt vertical mirror

f_2

Refl wrt SW-NE / pos slope







$\pi = 4\ 2\ 3\ 6\ 5\ 1\ 7$

$P :=$ insertion tableau of π

$Q :=$ recording tableau of π

π.reverse()

7 1 5 6 3 2 4

$P(\pi^{rev}) = P^{transpose}$

Sagan Thm 3.2.3 book

$Q(\pi^{rev}) = ev(Q^{transpose})$

Task Find this theorem in Sagan book Sec 3.9

π.inverse()

$\pi^{-1} = 6\ 2\ 3\ 1\ 5\ 4\ 7$

$P(\pi^{-1}) = Q(\pi)$

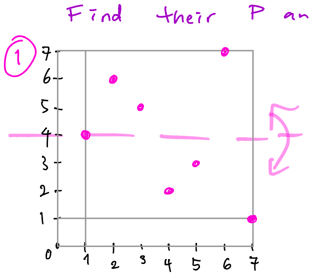
$Q(\pi^{-1}) = P(\pi)$

This Dihedral group is generated by R_1 and R_2, so the other five matrices are compositions of R_1 and R_2. This also means that the only tableaux that can appear are these eight tableaux:

P, Q, ev(P), ev(Q), and their conjugates

PRACTICE Find the other five permutations whose xy coord matrices are results of applying reflections / rotations to Mat(π). Find their P and Q tableaux.

(1)



π. complement ()

4 6 5 2 3 7 1

$P(\pi^{compl}) =$

$Q(\pi^{compl}) =$

(2) (3) (4) (5)

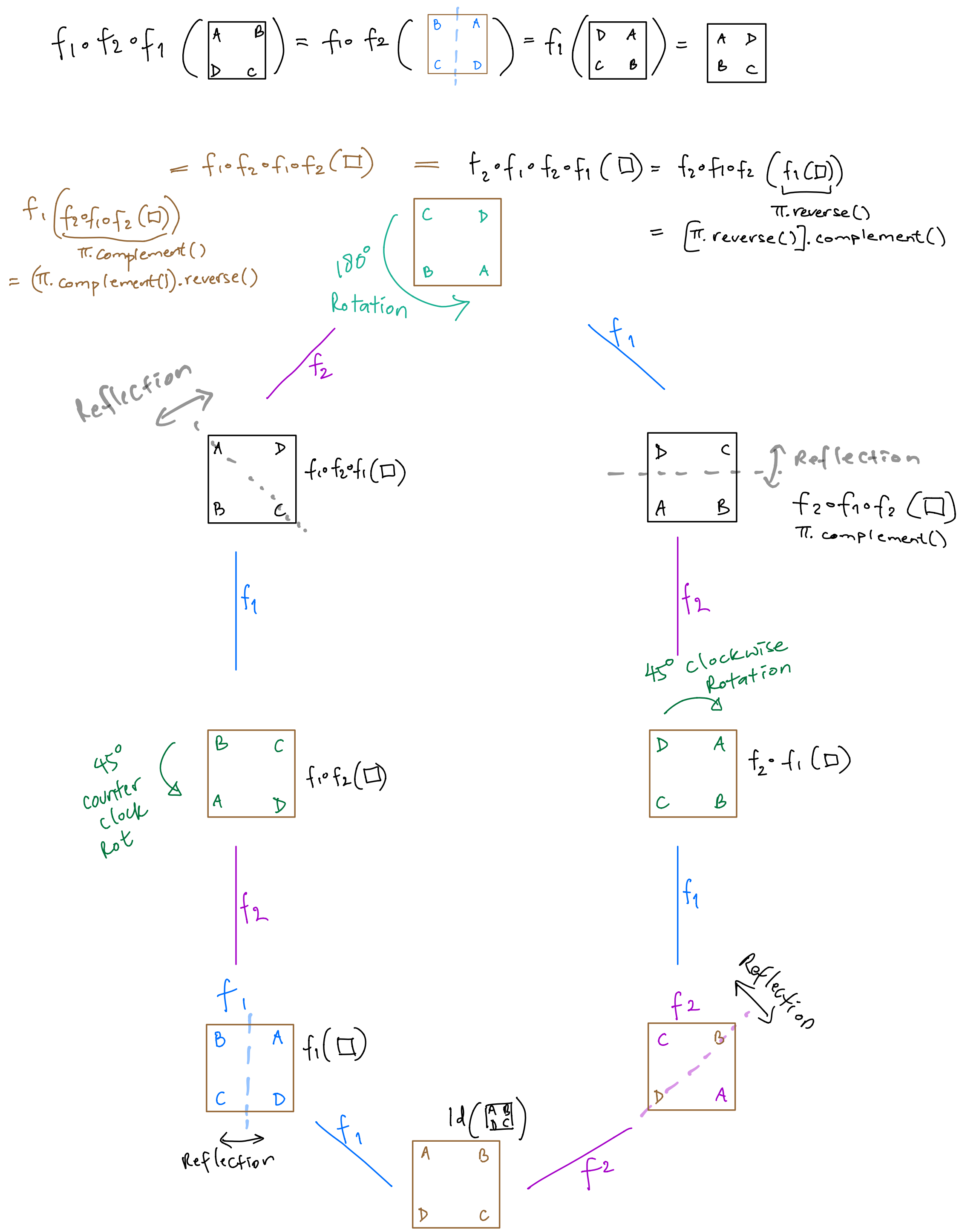
$P(\pi.\text{reverse}().\text{complement}()) = ev(P(\pi))$

Partial answers to PRACTICE on the next two pages

Dihedral group of the 8 symmetries of a square

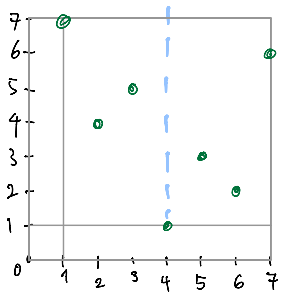
Edges correspond to multiplication (function composition) on the right.

Read the product from right to left (like function composition) as usual



$f_1 \circ f_2$ = 45° Rot counterclock

Refl wrt SW-NE / pos slope
then wrt vertical mirror



$(\pi.\text{inverse}()).\text{reverse}()$

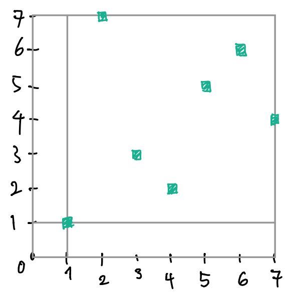
$(R_1 \circ R_2)(\pi)$ = 7 4 5 1 3 2 6

Insertion tableau $[Q(\pi)]^{\text{transpose}}$

Recording tableau ev $[P(\pi)^{\text{transpose}}]$

$f_1 \circ f_2 \circ f_1 \circ f_2 =$
$f_2 \circ f_1 \circ f_2 \circ f_1$

180° Rotation

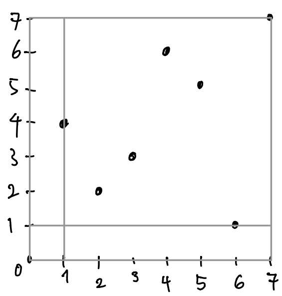


$R_1 \circ R_2 \circ R_1 R_2(\pi) = (\text{compl}(\pi))^{Rev} = \text{compl}(\pi^{Rev})$

1 7 3 2 5 6 4

Insertion tableau ev $[P(\pi)]$

Recording tableau ev $[Q(\pi)]$



1 2 3 4 5 6 7

π = 4 2 3 6 5 1 7

"Complement" $(P(\pi^{rev,\, comp})) =$ backward RSK

Exercise 15 Use Fukuda BBS.

- Prove that complement$\left(ev\left(P(\pi)\right)\right) =$ backward RSK (π).
- Think about a natural convention for backward RSK (π), for the purpose of connecting it to back Soliton Decomposition. (maybe follow the rule for writing the forward Soliton Decomposition)

Recall from Prob 2 part 3 notes

Prop Sec 1.9 (Corollary of LLPS paper) One Fukuda BBS move is the same as applying F_1, n times.

Applying F_1 means:

- Make ball ① jump to the first available empty cell.
- Change ① to ⓝ
- Then decrease the value of all the other balls by 1.

E.g. $\pi =$ 2 1 4 3

F_1: 1 _ 3 2 ④

F_1: ④ 2 1 3

F_1: 3 1 _ 2 ④

F_1: 2 _ 4 1 3 is the configuration after doing one (Fukuda) BBS move.

Task to try:

(i) Write the backward Fukuda BBS move version of Prop Sec 1.9.

(ii) Study the def of evacuation in Sagan Sec 3.9 (defined as sort of the opp of JdT).

Can you make a connection between this and the "revised Prop Sec 1.9"?

~ end lecture Thurs, Jun 25 ~

$w = 4\ 2\ 3\ 6\ 5\ 1\ 7$

Backward Sol Dec(w) =

possible convention #1
```
2 4 6 7
      3
      5
      1
```
or

possible convention #2 (= 180° Rotation of Convention #3)
```
      1
      5
      3
2 4 6 7
```
or

possible convention #3
```
7 6 4 2
3
5
1
```

```
t=-1        2 4 6 7 3     5   1
t=0   w=           4 2 3 6 5 1 7
t=1                4     6 2   1 3 5 7
```

Forward Sol Dec(w)
```
1 3 5 7
2
6
4
```

Let w^{rc} be the reverse - complement permutation of w

$w^{rc} = 1\ 7\ 3\ 2\ 5\ 6\ 4$

Backward Sol Dec(w^{rc}) = convention #1
```
1 3 5 7
      6
      2
      4
```

```
t=-1        1 3 5 7   6 2     4
t=0   w^rc =       1 7 3 2 5 6 4
t=1                  7 3     5 1 2 4 6
```

Forward Sol Dec(w^{rc}) =
```
1 2 4 6
5
3
7
```

- Compl(Forward Sol Dec(w^{rc})) =
```
7 6 4 2
3
5
1
```
= Convention #3 of Backward Sol Dec(w)

- Compl and reverse (Forward Sol Dec(w^{rc})) =
```
2 4 6 7
      3
      5
      1
```
= convention # 1 of Backward Sol Dec(w)

- 180° Rotation of compl(Forward Sol Dec(w^{rc})) [compl(Forward Sol Dec(w^{rc})) = Convention #3 of Backward Sol Dec(w)] =
```
      1
      5
      3
2 4 6 7
```
= convention #2 of Backward Sol Dec(w)

- compl & reverse (Forward Sol Dec(w)) =
```
1 3 5 7
      6
      2
      4
```
= convention #1 of Backward Sol Dec(w^{rc})

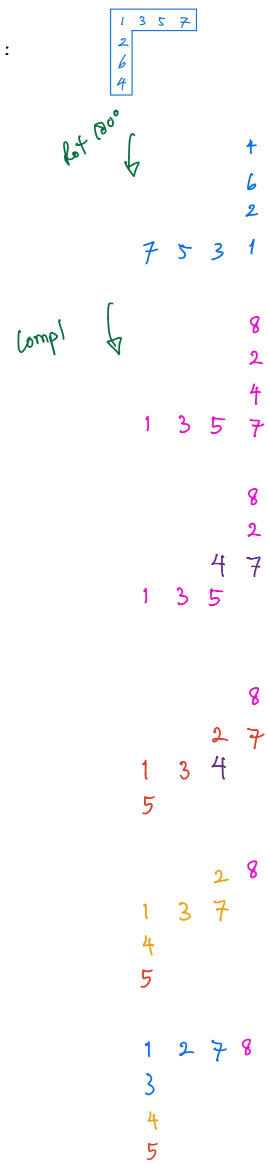
[Forward Sol Dec(w) =
```
1 3 5 7
2
6
4
```
]

Joel (June 2020) Remark:

(Fukuda)
Forward convention: balls jump $\xrightarrow{\text{left to right}}$ starting w/ smallest balls

The backward step is equivalent to the following:

- Subtract all ball numbers from $n+1$
- Reverse the direction of the line
- Do a forward step
- Subtract the numbers from $n+1$
- Reverse the direction of the line again



E.g.

$t=0$	0	0	0	2	0	1	3	4	0	0
Reverse &	0	0	4	3	1	0	2	0	0	0
complement	0	0	1	2	4	0	3	0	0	0
Forward Step	0	0	0	0	0	1	4	2	3	0
Reverse &	0	3	2	4	1	0	0	0	0	0
complement	0	2	3	1	4	0	0	0	0	0