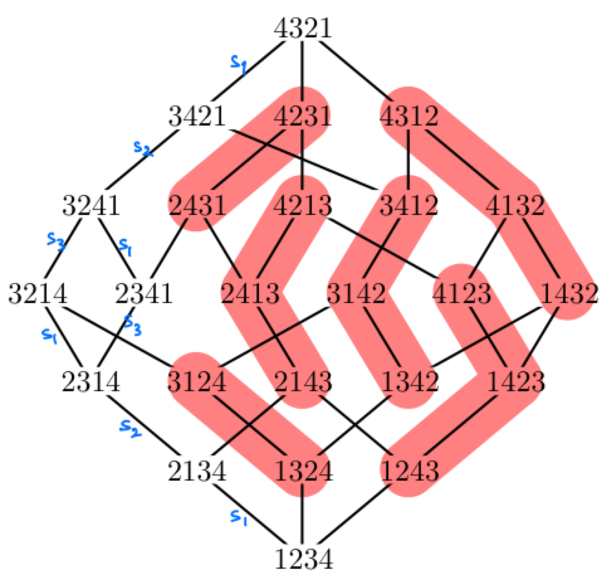
Last updated Thurs June 25, 2020

2.12 Commutation classes

Def/Fact Let $\pi \in S_{n+1}$. For two reduced words w_1, w_2 of π, say $w_1 \underset{\text{Comm}}{\sim} w_2$ if w_2 is obtained from w_1 by a sequence of commutation moves. Then $\underset{\text{Comm}}{\sim}$ is an equivalence relation. The equivalence classes are called <u>commutation classes</u> of π.
(also called <u>commutativity classes</u>)

Example: If the Coxeter element is $c = s_1 s_2 s_3$, The c-sorting word for $w_0 = 4321$ is $s_1 s_2 s_3 s_1 s_2 s_1$.

The commutation class containing $s_1 s_2 s_3 s_1 s_2 s_1$ contains only one other reduced word, $s_1 s_2 s_1 s_3 s_2 s_1$.

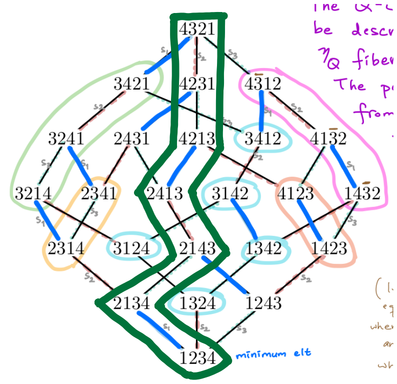


Example: If the Coxeter elt is $c = s_1 s_3 s_2$

$c^{\infty} = s_1 s_3 s_2 \,|\, s_1 s_3 s_2 \,|\, s_1 s_3 s_2 \,|\, s_1 s_3 s_2 \,|\, \cdots$

The (s_1, s_3, s_2)-sorting word for 4321 is $s_1 s_3 s_2\, s_1\, s_3 s_2$

$s_1 s_3 s_2 \,|\, s_1 s_3 s_2 \,|\, s_1 s_3 s_2 \,|\, s_1 s_3 s_2 \,|\, \cdots$



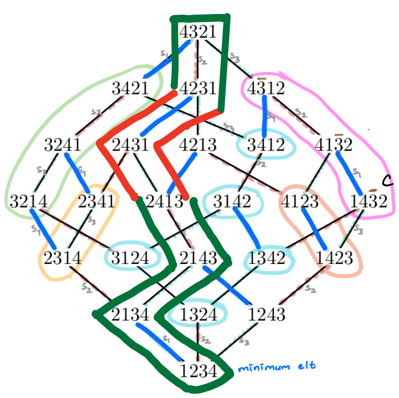
There are many reduced words for w_0 (counted using tableaux of ⊞), but there are only four reduced words for w_0 in the commutation class containing the c-sorting word

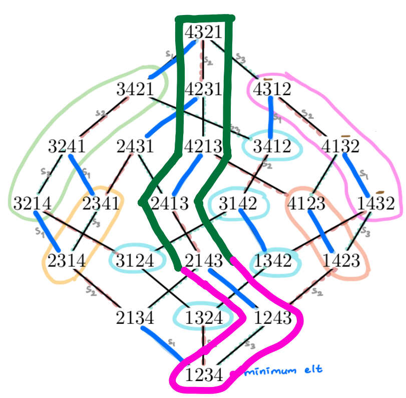
$s_1 s_3 s_2\, s_1\, s_3 s_2$

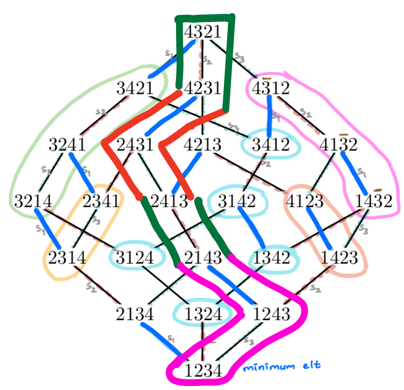
$s_1 s_3 s_2\, s_3 s_1\, s_2$

$s_3 s_1 s_2\, s_1\, s_3 s_2$

$s_3 s_1 s_2\, s_3 s_1\, s_2$.

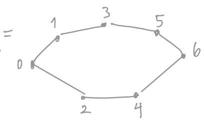






Example

$n = 4$

$Q =$ 

The commutation class containing the c-sorting word of w_0, for $c = s_1 s_3 s_2 s_4$

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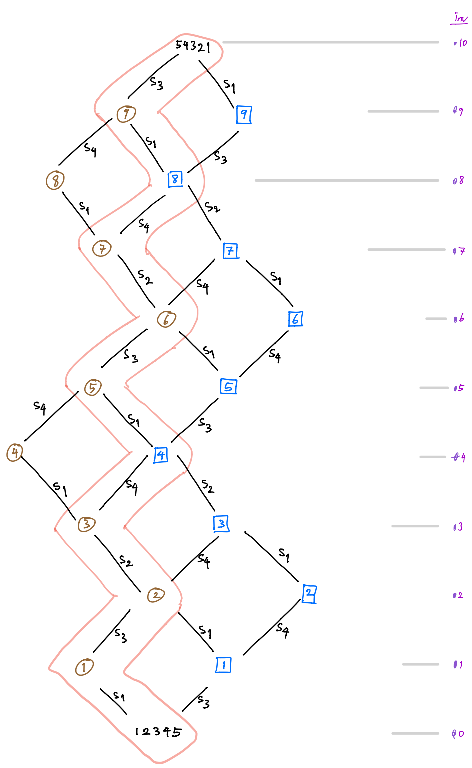
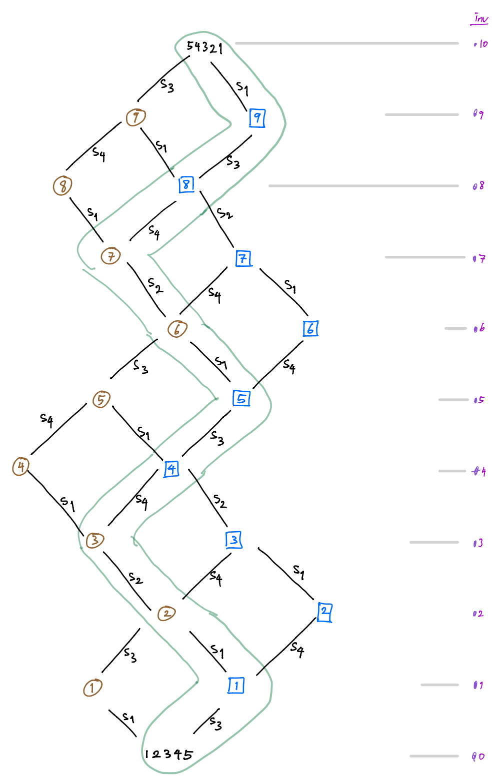
Example

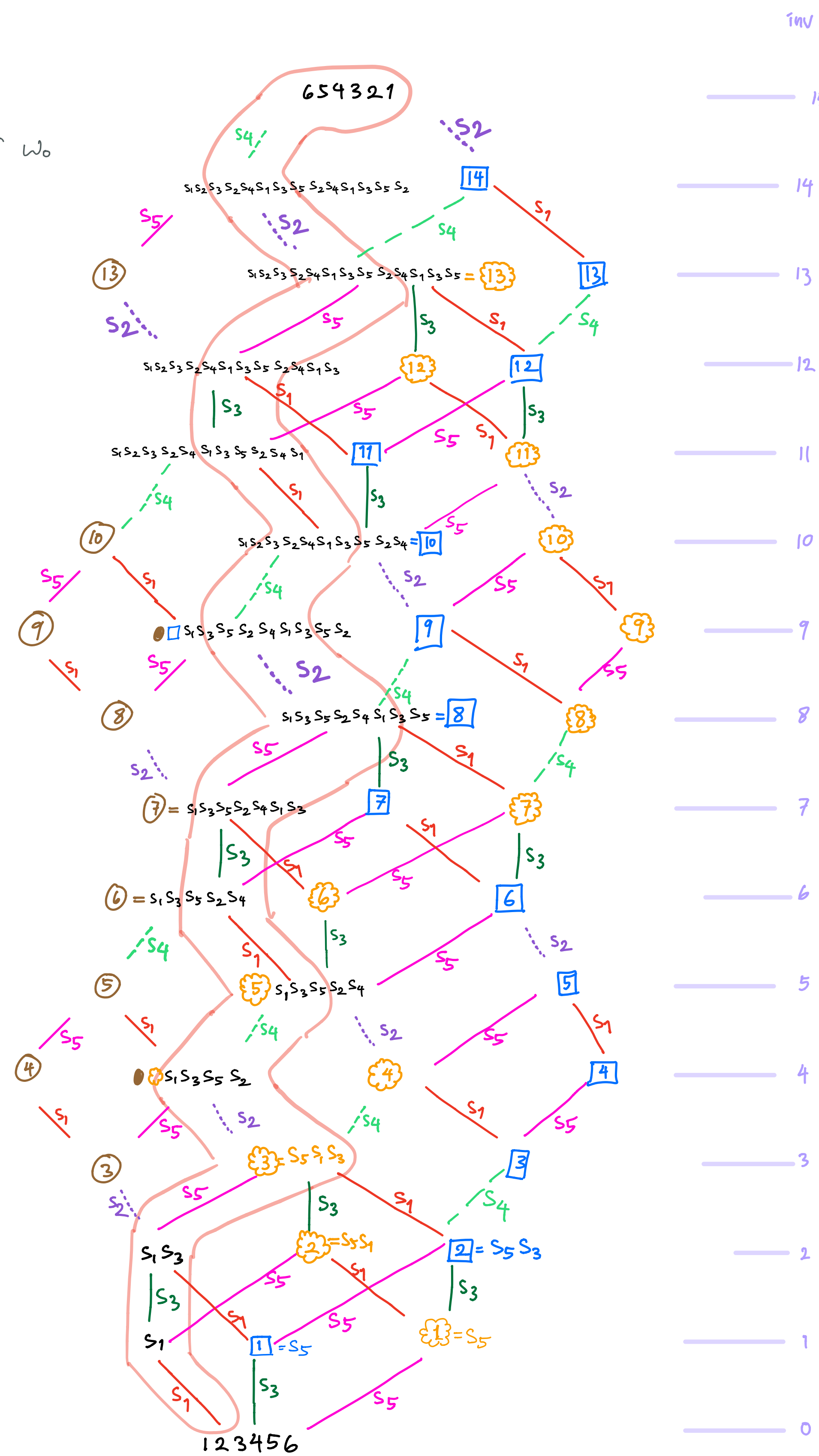
The commutation class containing the c-sorting word of w_0, for $c = s_3 s_1 s_2 s_4$





Commutation class containing the c-sorting word of w_0 for

$c = s_1 s_3 s_5 s_2 s_4$



Commutation class containing the c-sorting word of w_0 for

$c = s_5 s_3 s_1 s_2 s_4$

is the same commutation class as for the $(s_1, s_3, s_5, s_2, s_4)$-sorting word of w_0

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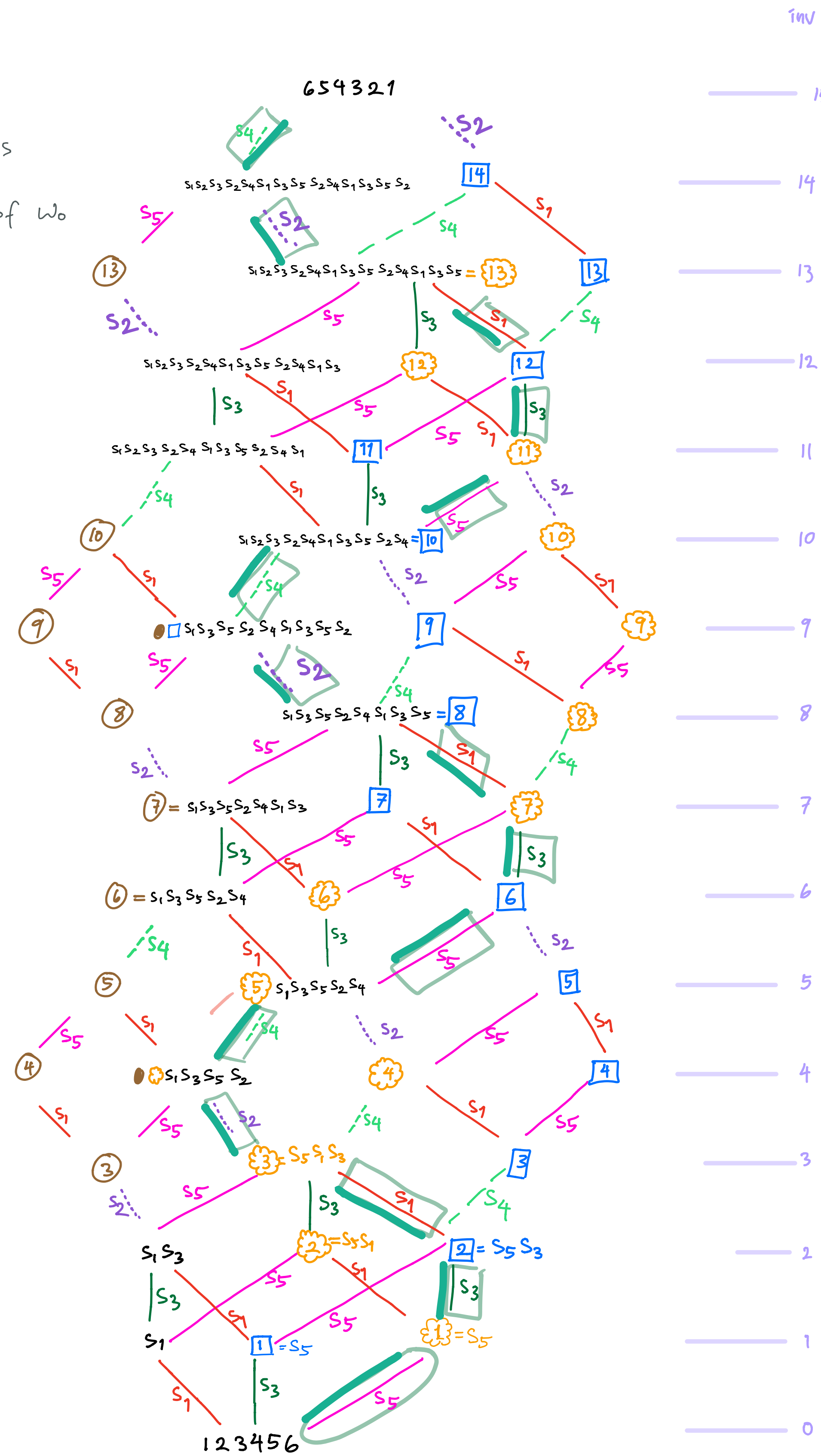
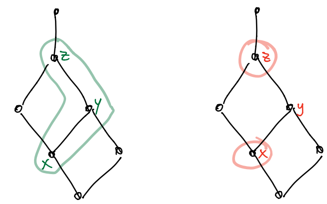


2.13 Commutatation class and saturated chains of c-sortable words

Def

A <u>chain</u> in a poset is <u>saturated</u> if no element can be added between two of its elements without losing the property of being a chain.



a saturated chain　　a non-saturated chain

Lemma 2.13 Fix a Coxeter element $c = a_1 a_2 \cdots a_n$.

Suppose π is a c-sortable elt with c-sorting word $u_1 u_2 \cdots u_\ell$.

① Then each prefix of $u_1 u_2 \cdots u_\ell$,

$$\begin{array}{l} Id, \\ u_1, \\ u_1 u_2, \\ u_1 u_2 u_3, \\ \vdots \\ u_1 u_2 u_3 \cdots u_{\ell-1}, \\ u_1 u_2 u_3 \cdots u_{\ell-1} u_\ell \end{array}$$

is a c-sortable word (that is, the c-sorting word of a c-sortable element).

In other words, $u_1 u_2 \cdots u_\ell$ gives a length-ℓ saturated chain (in the weak order) of c-sorting words of c-sortable elements $Id, u_1, u_1u_2, \cdots, u_1 u_2 u_3 \cdots u_\ell$.

Pf Follows from def of c-sorting words and c-sortable elements.

② Suppose there are i, j where $|i-j| \geqslant 2$ and $u_k = s_i$, $u_{k+1} = s_j$ for some $k \in [\ell-1]$.

[Recall that $u_1 u_2 \cdots u_{k+1} u_k \cdots u_\ell$ is another reduced word of π (by the commutation relation).]

Then every prefix of $u_1 u_2 \cdots u_{k+1} u_k \cdots u_\ell$,

$$\begin{array}{l} Id, \\ u_1, \\ u_1 u_2, \\ \vdots \\ u_1 u_2 \cdots u_{k-1}, \\ u_1 u_2 \cdots u_{k-1} u_{k+1}, \\ u_1 u_2 \cdots u_{k-1} u_{k+1} u_k, \\ \cdots \\ u_1 u_2 \cdots u_{k-1} u_{k+1} u_k \cdots u_\ell \end{array}$$

is a reduced word of a c-sortable permutation (although in general not the c-sorting of the corresp. perm).

So these prefixes give a length-ℓ saturated chain of c-sortable elements.

Pf
- The first k prefixes of $u_1 u_2 \cdots u_{k+1} u_k \cdots u_\ell$ are precisely the first k prefixes of $u_1 u_2 \cdots u_k u_{k+1} \cdots u_\ell$.
- By the commutation relation, $u_1 u_2 \cdots u_{k+1} u_k \cdots u_j = u_1 u_2 \cdots u_k u_{k+1} \cdots u_j$ (as permutation) for each prefix of $u_1 u_2 \cdots u_{k+1} u_k \cdots u_\ell$ of length $k+1$ or longer.
- It remains to show that $u_1 u_2 \cdots u_{k-1} u_{k+1}$ is a c-sortable permutation.

(3) Suppose $v_1 v_2 \cdots v_\ell$ is another reduced word of π which can be obtained by a non trivial sequence of short braid moves (commutation) ($s_i s_j = s_j s_i$ if $|i-j| \geq 2$).

Then the word $v_1 v_2 \cdots v_\ell$ is not the c-sorting word of π (since not all $u_k = v_k$).

But all prefixes of $v_1 v_2 \cdots v_\ell$,

$$\begin{gathered} Id, \\ v_1, \\ v_1 v_2, \\ v_1 v_2 v_3, \\ \vdots \\ v_1 v_2 v_3 \cdots v_{\ell-1}, \\ v_1 v_2 v_3 \cdots v_{\ell-1} v_\ell, \end{gathered}$$

are reduced words of c-sortable elements.
(in general, their c-sorting words are not equal to these prefixes.)

The word $v_1 v_2 \cdots v_\ell$ gives a different length-ℓ saturated chain of c-sortable elements $Id, v_1, v_1 v_2, \ldots, v_1 v_2 \cdots v_\ell$.
(In general, their c-sorting words are not equal to these words) $\boxed{\text{Lemma 2.12}}$

REU Exercise 12

Finish proving part (2) of Lemma.
Prove part (3) of Lemma.

Example 1 of Lemma

Let $c = s_1 s_2 s_3 s_4 s_5$ (Tamari Coxeter elt)

Let $\pi := s_1 s_3 s_4 s_5 s_3 s_4$

To check that $s_1 s_3 s_4 s_5 s_3 s_4$ is a reduced work, I find the one-line notation of π (215643) then verify $inv(\pi) = 6$.

$s_1\ s_2\ s_3\ s_4\ s_5 \,|\, s_1\ s_2\ s_3\ s_4\ s_5$

is also the c-sorting word of π, which shows that π is c-sortable.

$s_1 s_3 s_4 s_5 s_3 s_4 = 2\ 1\ 5\ 6\ 4\ 3$ $\quad inv = 6$

↑ Multiply on right by s_4

$s_1 s_3 s_4 s_5 s_3 = 2\ 1\ 5\ 4\ 6\ 3$

↑ Multiply on right by s_3

$s_1 s_3 s_4 s_5 = 2\ 1\ 4\ 5\ 6\ 3$

↑ Multiply on right by s_5

$s_1 s_3 s_4 = 2\ 1\ 4\ 5\ 3\ 6$

↑ Multiply on right by s_4

$s_1 s_3 = 2\ 1\ 4\ 3\ 5\ 6$

↑ Multiply on right by s_3

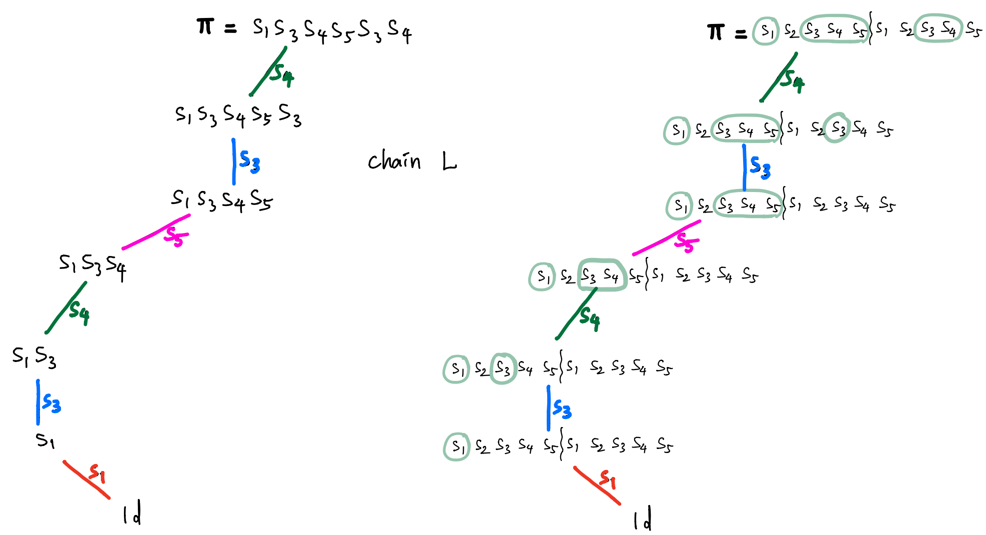
$s_1 = 2\ 1\ 3\ 4\ 5\ 6$

↑ Multiply on right by s_1

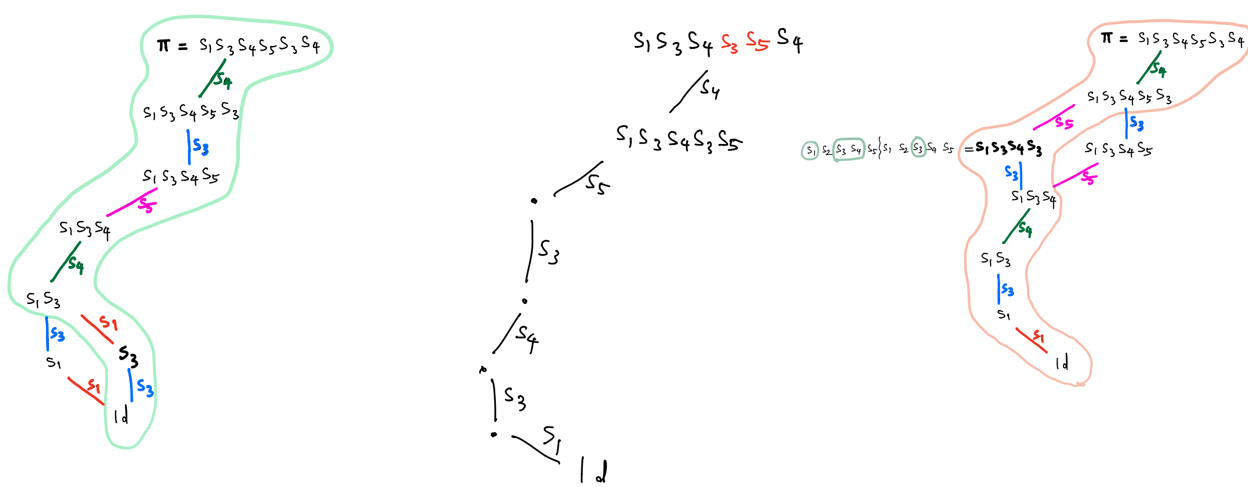
$Id = 1\ 2\ 3\ 4\ 5\ 6$

(Con't) Example 1 of Lemma

① Part 1 of Lemma tells me that I have a saturated length-6 chain L of c-sortable elements from Id to π in the weak order :

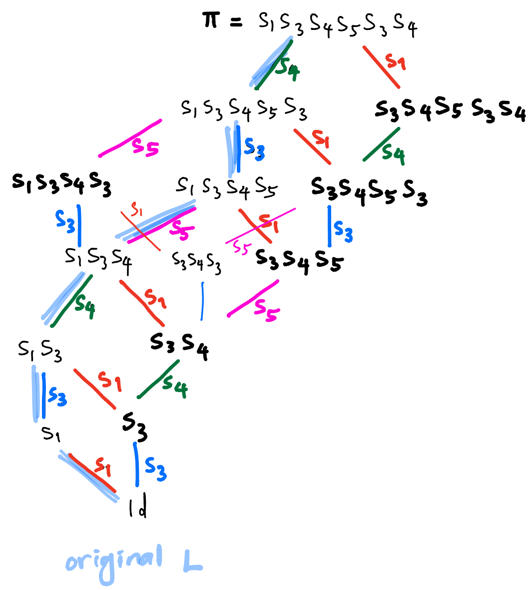


② Applying Part 2 of Lemma once to L would produce another saturated length-6 chain of c-sortable elements from Id to π in the weak order by swapping s_1, s_3 or s_5, s_3 in the chain L.



(Con't) Example 1 of Lemma

③ Applying part 3 of Lemma to L gives a total of twelve saturated length-6 chains of c-sortable elements from Id to π in the weak order.



Connection of linear extensions of heaps, learn later if needed.

Heap of $s_1 s_3 s_4 s_5 s_3 s_4$

$u_1\ u_2\ u_3\ u_4\ u_5\ u_6$

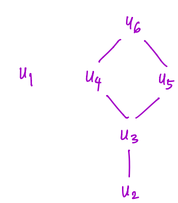
u_1

$u_2 \sqsubset u_3, u_5, u_6$

$u_3 \sqsubset u_4, u_5, u_6$

$u_4 \sqsubset u_6$

$u_5 \sqsubset u_6$



Linear extensions:

$u_2\ u_3\ u_4\ u_5\ u_6$

$u_2\ u_3\ u_5\ u_4\ u_6$

Put u_1 at any of the six possible positions

total of twelve linear extensions 2×6

PRACTICE

Pick a saturated chain L in the weak order of c-sortable words from one of the examples above.

Starting from the chain L, apply a long braid move $s_j s_i s_j \to s_i s_j s_i$ where $|i-j|=1$ to produce a new saturated chain L' in the weak order.

L' has the same length as L, but not all elements of L' are c-sortable.

Which elements in L' are c-sortable and which elements are not?

REU Exercise 13 (put in the same section as REU Exercise 12)

Write a new Lemma/proposition: If L is a saturated chain of c-sortable words in the weak order, describe the new saturated chain L' (in the weak order) which you get by applying a long braid move $s_i s_j s_i \to s_j s_i s_j$ where $|i-j|=1$.

Prop

Let $\pi \in S_{n+1}$. Let ℓ be the length of π.

The reduced words of π are in bijection with (saturated) length-ℓ chains in the weak order from Id to π.

Given a reduced word $u_1 u_2 \dots u_\ell$ of π,

it is sent to the chain $Id \xrightarrow{u_1} u_1 \xrightarrow{u_2} u_1 u_2 \xrightarrow{u_3} \dots \xrightarrow{u_\ell} \pi$.

Pf This follows from the def/theory of the weak order.

Note: Often people write \mathcal $R(\pi)$ to denote the set of reduced words of π.

Prop

Let $\pi \in S_{n+1}$. Let ℓ be the length of π.
Let L be a (saturated) length-ℓ chain in the weak order from Id to π.

Then each (saturated) length-ℓ chain in the weak order from Id to π is obtained by a sequence of short braid (commutation) moves and long braid moves from the chain L.

Pf This follows from the relations of between the simple reflections $s_1, \dots, s_n$.

Application of Lemma part (3) to Problem II:

Recall that we can think of a c-Cambrian lattice as a poset on c-sortable elements (by Thm 2.11 (b)).

Proposition 2.13 (Corollary of Lemma (3))

Let w be the c-sorting word of the longest element w_0 in S_{n+1}.
The reduced words in the commutation class of w are in bijection with maximum-length chains in the c-Cambrian lattice:

Given a reduced word $w_0 = u_1 u_2 \cdots u_{\binom{n+1}{2}}$ in the commutation class, it is sent to the chain $\mathrm{Id} \xrightarrow{u_1} u_1 \xrightarrow{u_2} u_1 u_2 \xrightarrow{u_3} \cdots \xrightarrow{u_{\binom{n+1}{2}}} w_0$.

REU Exercise 14 (put in the same section as REU Exercise 12 and 13)

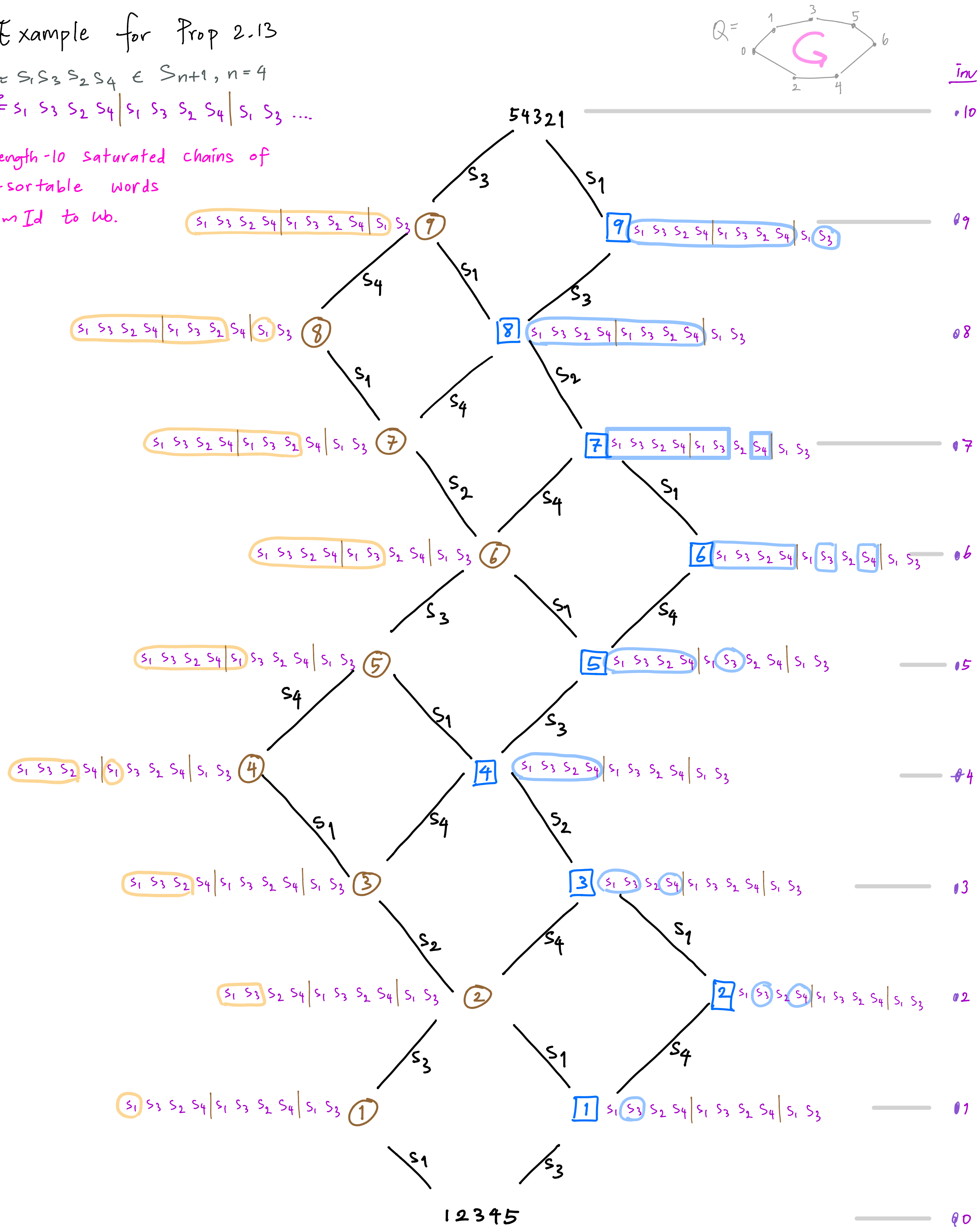
Use the lemmas and propositions in this section 2.13 to prove the above Proposition 2.13.

Example for Prop 2.13

$c = s_1 s_3 s_2 s_4 \in S_{n+1}, n=4$

$c^\infty = s_1\ s_3\ s_2\ s_4 \mid s_1\ s_3\ s_2\ s_4 \mid s_1\ s_3 \ldots$

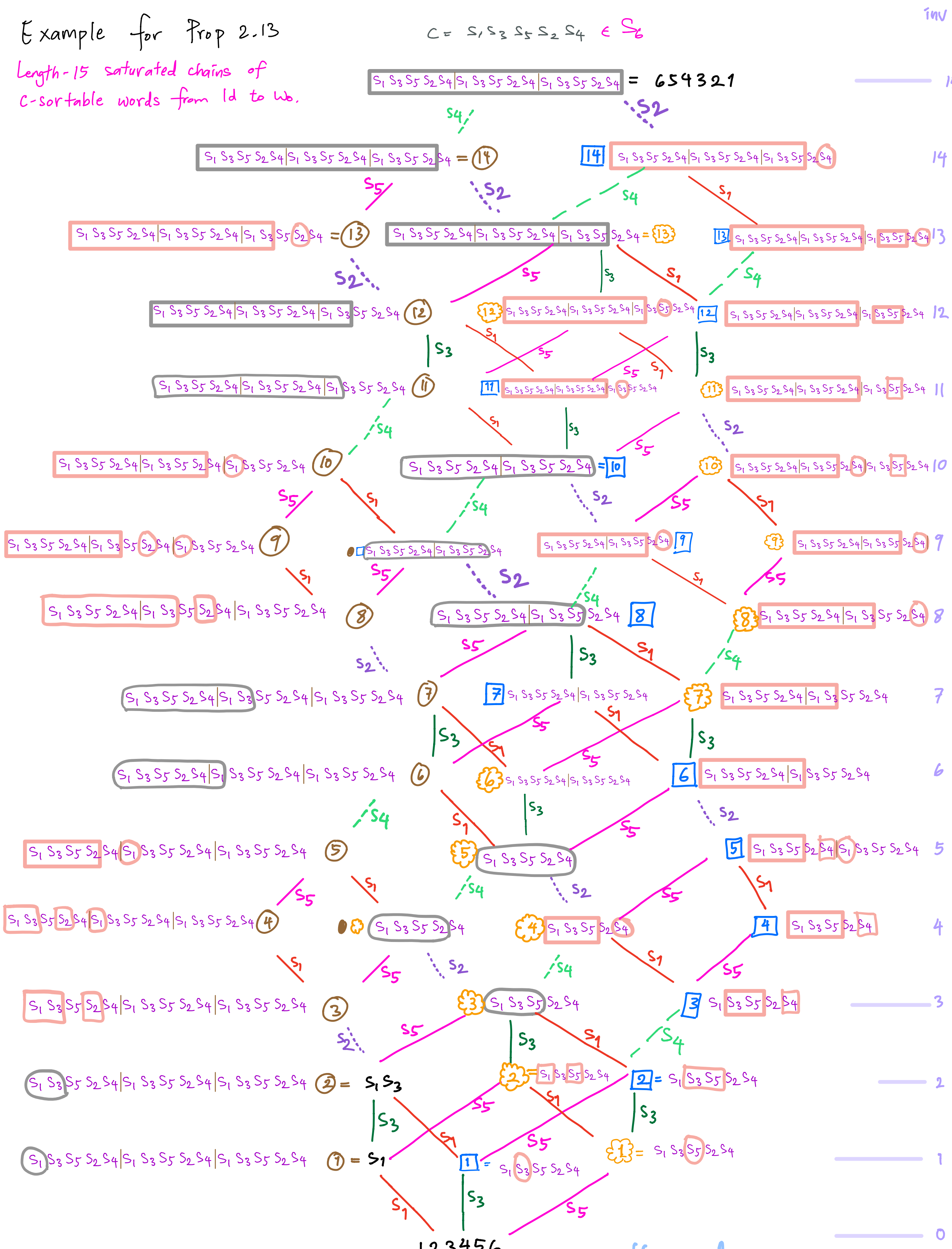
Length-10 saturated chains of c-sortable words from Id to wo.



Example for Prop 2.13

$c = s_1 s_3 s_5 s_2 s_4 \in S_6$

Length-15 saturated chains of c-sortable words from Id to w_0.



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