ADVICE ON MATHEMATICAL WRITING

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This handout lists some writing tips when you are preparing a mathematical text.

1. Notation

(1) Do not begin sentences with a symbol.

Bad: x is positive, so it has a square root.
Good: Since x is positive, it has a square root.

Bad: Let n be an even number. $n = 2m$ for some $m \in \mathbf{Z}$.
Good: Let n be an even number. Thus $n = 2m$ for some $m \in \mathbf{Z}$.
Good: Let n be an even number, so $n = 2m$ for some $m \in \mathbf{Z}$.

Bad: One solution is $f(x) = \sin x$. $f(x)$ is periodic.
Good: One solution is $f(x) = \sin x$. In this case, $f(x)$ is periodic.

(2) If two mathematical symbols are not part of the same mathematical expression, they should never appear next to each other with no words or grammatical marks in between them.

Bad: If $n \neq 0$ $n^2 > 0$.
Good: If $n \neq 0$, $n^2 > 0$.
Good: If $n \neq 0$ then $n^2 > 0$.

(3) When introducing notation, make it fit the context. A lot of the time a choice of notation is just common sense.

Bad: Let m be a prime.
Good: Let p be a prime.

Bad: Let X be a set, and pick an element of X, say t.
Good: Let X be a set, and pick an element of X, say x.

Bad: Pick two elements of the set X, say x and u.
Good: Pick two elements of the set X, say x and y.
Good: Pick two elements of the set X, say x_1 and x_2.
Good: Pick two elements of the set X, say x and x'.

(4) Always define new notation (is it a number? a function? of what type?) and be clear about its logical standing.

Very bad: Since n is composite, $n = ab$.
Bad: Since n is composite, $n = ab$ for some integers a and b.
Good: Since n is composite, $n = ab$ for some integers a and b greater than 1. [*Every* integer is a product, since $n = n \cdot 1$, so writing $n = ab$ alone introduces no constraint whatsoever.]

Bad: If a polynomial $f(x)$ satisfies $f(n) \in \mathbf{Z}$, does $f(x)$ have integer coefficients?

Good: If a polynomial $f(x)$ satisfies $f(n) \in \mathbf{Z}$ for every $n \in \mathbf{Z}$, does $f(x)$ have integer coefficients?

(5) Do not give multiple meanings to a variable in a single argument.

Bad: To show the sum of two even numbers is even, suppose a and b are even. Then $a = 2m$ and $b = 2m$, for some integer m. We have $a+b = 4m = 2(2m)$, which is even. [Notice this reasoning shows the sum of any two even numbers is always a multiple of 4, which is nonsense.]

Good: To show the sum of two even numbers is even, suppose a and b are even. Then $a = 2m$ and $b = 2n$, for some integers m and n. We have $a + b = 2m + 2n = 2(m + n)$, which is even.

(6) Avoid overloading meaning into notation.

Bad: Let $x > 0 \in \mathbf{Z}$.
Good: Let x be an integer, with $x > 0$.
Good: Let x be a positive integer.

(7) **NEVER** use the logical symbols $\forall$ (for all), $\exists$ (there exists), $\wedge$ (and), and $\vee$ (or) when writing, *except* in a technical paper on logic. Write out what you mean in ordinary language.

Bad: The conditions imply $a = 0 \wedge b = 1$.
Good: The conditions imply $a = 0$ and $b = 1$.

Bad: If $\exists$ a root of the polynomial then there is a linear factor.
Good: If there is a root of the polynomial then there is a linear factor.

(8) The symbol $\forall$ means "For all" or "For every", not "All" or "Every".

Bad: $\forall$ square matrices with nonzero determinant are invertible.
Good: All square matrices with nonzero determinant are invertible.

Bad: In the complex plane $\forall$ complex number has a square root.
Good: In the complex plane every complex number has a square root.

Bad: If the functions agree at three points, they agree at $\forall$ points.
Good: If the functions agree at three points, they agree at all points.

(9) Avoid silly abbreviations, or the misuse of standard notations. or the use of abbreviations which are used strictly on the blackboard (like WLOG, s.t., and iff).

Bad: When n is $\int$, $2n$ is an even number.
Good: When n is integral, $2n$ is an even number.
Good: When n is an integer, $2n$ is an even number.

Bad: Let z be a $\mathbf{C}$.
Good: Let z be a complex number.
Good: Choose $z \in \mathbf{C}$.

Bad: WLOG, we can assume $x > 0$.
Good: Without loss of generality, we can assume $x > 0$.

Bad: There is a point x s.t. $f(x) > 0$.
Good: There is a point x such that $f(x) > 0$.

(10) If a piece of notation is superfluous in your writing, don't use it.

Bad: Every differentiable function f is continuous.
Good: Every differentiable function is continuous.
Good: All differentiable functions are continuous.

Bad: A square matrix A is invertible when its determinant is not 0.
Good: A square matrix A is invertible when $\det A \neq 0$.
Good: A square matrix is invertible when its determinant is not 0.

The difference between the use of A in the Bad example and in the first Good example above is that in the first Good example something is actually done with A: we refer to it again in $\det A$. In the Bad example the use of A is superfluous notation.

2. Equations and expressions

(1) If an equation or expression is important (either for its own sake or because you will refer back to it later), display the equation on its own line. If you need to refer to it later, label it (as (1), (2), and so on) on the side. Of course, if you only need to make a reference to a displayed equation or expression immediately before or after it appears, you could avoid a label and say "by the above equation," *etc.*

Bad: As a special case of the binomial theorem,

$$(x+y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4.$$

[suppose several lines of text are here]
By the equation 8 lines up, we see...
Good: As a special case of the binomial theorem,

$$(x+y)^4 = x^4 + 4x^3y + 6x^2y^2 + 4xy^3 + y^4. \tag{2.1}$$

[suppose several lines of text are here]
By equation (2.1), we see...

(2) If a single computation involves several steps, especially more than two, present the steps in stacked form.

Bad:

$(x+1)^3 = (x+1)^2(x+1) = (x^2+2x+1)(x+1) = x^3+3x^2+3x+1.$

Bad:

$$\begin{aligned} (x+1)^3 &= (x+1)^2(x+1) \\ (x+1)^3 &= (x^2+2x+1)(x+1) \\ (x+1)^3 &= x^3+3x^2+3x+1. \end{aligned}$$

Good:

$$\begin{aligned} (x+1)^3 &= (x+1)^2(x+1) \\ &= (x^2+2x+1)(x+1) \\ &= x^3+3x^2+3x+1. \end{aligned}$$

(3) Equations do not stand by themselves. They appear as part of a sentence and should be punctuated accordingly! If an equation ends a sentence, place a period at the end of the line. If an equation appears in the middle of a sentence, use a comma after the equation if one would naturally pause there. Sometimes no punctuation is needed after the equation.

Bad: We call x_0 a *critical point* of f when f is differentiable and

$$f'(x_0) = 0$$

.

Good: We call x_0 a *critical point* of f when f is differentiable and

$$f'(x_0) = 0.$$

Good: When f is differentiable, and x_0 satisfies

$$f'(x_0) = 0,$$

we call x_0 a *critical point*.

Good: When f is differentiable, any x_0 where

$$f'(x_0) = 0$$

is called a *critical point*.

(That the equation is displayed separately in each case simply serves to highlight its importance to the reader. It could have been included within the main text, and punctuation rules of course apply in the same way. The words "critical point" were set in italics to emphasize that this particular term is being defined. Some books put defined terms in **bold** in the definitions.)

3. Parentheses and Commas

(1) Avoid pointless parentheses in mathematical expressions.

Bad: $(x + y)(x - y) = (x^2 - y^2)$. [The parentheses on the right have no purpose.]
Good: $(x + y)(x - y) = x^2 - y^2$.

Bad: If 7 is a factor of the product $(a_1 a_2 \cdots a_n)$, then ...
Good: If 7 is a factor of the product $a_1 a_2 \cdots a_n$, then ...

Bad: The length is a factor of $(p - 1)$.
Good: The length is a factor of $p - 1$.

Bad: The Taylor series for $\log(1 + x)$ is

$$\sum_{n=1}^{\infty} \frac{(-1)^{(n-1)}}{n}.$$

Good: The Taylor series for $\log(1 + x)$ is

$$\sum_{n=1}^{\infty} \frac{(-1)^{n-1}}{n}.$$

Good: $(a + b)^2 - (a + c)^2 = b^2 - c^2 + 2ab - 2ac$.
Good: $(a + b)^2 - (a + c)^2 = (b^2 - c^2) + 2ab - 2ac$. [This example is good *only* if the writer wants the reader to view $b^2 - c^2$ as a single part of the right side.]

(2) Use parentheses to avoid confusing the meaning between a subtraction sign and a negative sign in a mathematical expression.

Very bad: $(a + b) - c = -ac - bc$. [If you look at the right side, you can see the writer meant for the left side to be the product of $a + b$ and $-c$, but the left side instead looks like "a plus b minus c."
Bad: $(a + b) \cdot -c = -ac - bc$.
Good: $(a + b)(-c) = -ac - bc$.

(3) Commas are natural places to pause briefly, but not as fully as a period. If you read something in your head, you should be able to notice badly placed commas, either because no pause should occur or because a period should be there instead of a comma.

Bad: The condition we want is, $a = 2b$.
Good: The condition we want is $a = 2b$.

Bad: The set is infinite, we pick a large finite subset of it.
Good: The set is infinite. We pick a large finite subset of it.

(4) While "If ..., then..." is a common phrase, it is bad English to write "Let..., then...." with a comma as the separator.

Very Bad: Let n be an even number, then $n = 2m$ for some $m \in \mathbf{Z}$.
Good: Let n be an even number. Then $n = 2m$ for some $m \in \mathbf{Z}$.
Good: Let n be an even number, so $n = 2m$ for some $m \in \mathbf{Z}$.

4. Use helpful words

(1) Tell the reader where you are going.

Good: We will prove this by induction on n.
Good: We will prove this by induction on the dimension.
Good: We argue by contradiction.
Good: Now we consider the converse direction.
Good: But $f(x)$ is actually continuous. To see why, consider...
Good: The inequality $a \leq b$ is strict: $a < b$. Indeed, if there was equality then...

(2) Use key words to show the reader how you are reasoning. These include

since, because, on the other hand, observe, note.

At the same time, vary your choice of words to avoid monotonous writing. This may require you to completely rewrite a paragraph.

Bad: We proved, for any a, that if a^2 is even, then a is even. Now suppose a^8 is even. Since $a^8 = (a^4)^2$, we obtain that a^4 is even. Then a^2 is even. Then a is even.
Good: We proved, for any a, that if a^2 is even, then a is even. Now suppose a^8 is even. Then, by successively applying the result we proved to a^4, a^2, and a, we see that a is even.

(3) Watch your spelling! If you aren't sure of the difference between "necessary" and "neccessary" or "discriminate" and "discriminant," look it up. (Canadian students may use their own flavour of spelling, but non-native English speakers should be careful not to let the grammatical rules of their native language affect their writing in English where those rules are different.)

Use "it's" only to mean "it is". The word "its", like "his" and "her", refers to possession.

Bad: It's clear that $f(x)$ has a real root since it's degree is odd.
Bad: Its clear that $f(x)$ has a real root since its degree is odd.
Good: It's clear that $f(x)$ has a real root since its degree is odd.
Good: Since $f(x)$ has odd degree, clearly it has a real root. [Write like this if you can't remember the difference between its and it's.]

Bad: Its surely true that starting your final draft on the last day will leave its mark in your work.
Good: It's surely true that starting your final draft on the last day will leave its mark in your work.

5. Types of Mathematical Results

In mathematics, results are labelled as either a theorem, lemma, or corollary. What's the difference?

- A theorem is a main result.
- A lemma is a result whose primary purpose is to be used in the proof of a theorem but which, on its own, is not considered significant or as interesting.
- A corollary is a result that follows from a theorem. It could be a special case of the theorem or a particularly important consequence of it.

So theorems stand on their own, a lemma always comes before a theorem, and corollaries always come after a theorem. The order in which these appear, then, is always

Lemma, Theorem, Corollary.

There is no reason a theorem must have a lemma before it or a corollary after it. But if you have a string of lemmas which don't lead to a theorem, for instance, then it will look strange to anyone experienced with mathematical writing.

Here are two examples. First we give a lemma and a theorem whose proof depends on the lemma.

Lemma 5.1. *In the integers, if d is a factor of a and b then d is a factor of $ax + by$ for any integers x and y.*

Proof. Since d is a factor of both a and b, we can write $a = dm$ and $b = dn$ for some integers m and n. Then for any x and y we have

$$ax + by = dmx + dny = d(mx + ny),$$

which shows d is a factor of $ax + by$. □

Theorem 5.2. *If a and b are integers and $ax_0 + by_0 = 1$ for some integers x_0 and y_0, then a and b have no common factor greater than 1.*

Proof. This will be a proof by contradiction. Suppose there is a common factor $d > 1$ of a and b. Applying Lemma 5.1 to the particular combination $ax_0 + by_0$, d is a factor of $ax_0 + by_0$, so d is a factor of 1. But there are no factors of 1 which are greater than 1, so we have a contradiction. Therefore a and b have no common factor greater than 1. □

Next we give a theorem in linear algebra and a corollary which follows from the theorem.

Theorem 5.3. *For any two square matrices A and B, $\det(AB) = (\det A)(\det B)$.*

[The proof of this is involved and is not included here.]

Corollary 5.4. *An invertible matrix has a nonzero determinant.*

Proof. If A is invertible, say of size $n \times n$, then $AB = I_n$ for some matrix B. Taking the determinant of both sides, Theorem 5.3 tells us

$$(\det A)(\det B) = \det(I_n) = 1,$$

so $\det A \neq 0$. □

Why do we need lemmas at all? Could we call everything a theorem? Yes, but the point of the three different names (lemma, theorem, corollary) is to indicate to the reader how the writer views the comparative standing of the different results.

Although lemmas are principally intended to be used for the proof of a more important result, sometimes a lemma turns out to be a very significant

In addition to the mathematical results in a paper, terminology is used and may need to be defined for the reader. Remember that a definition is not a theorem or anything like that. It's just a description of a new word. Here are two definitions.

Definition 5.5. A *geodesic* is a curve that locally minimizes lengths between points.

Definition 5.6. When all the elements in a partially ordered set are comparable to each other, we call it a *totally ordered* set.

Here is a bad definition.

Definition 5.7. If two sequences $\{a_n\}$ and $\{b_n\}$ converge to A and B, respectively, then the *limit* of the sequence $\{a_n + b_n\}$ is $A + B$.

Why isn't this a definition? Because saying that $\lim_{n\to\infty}(a_n + b_n) = A + B$ is a *result* which needs an explanation. In other words, this definition should be a theorem.

6. Fonts

Here are a few points about italic and non-italic fonts in mathematical writing:

- Single letters variables are set in italics: a, b, $a+bi$, x, and y. The quadratic formula is not
$$\frac{-\mathrm{b} \pm \sqrt{\mathrm{b}^2 - 4\mathrm{ac}}}{2\mathrm{a}}$$
but rather
$$\frac{-b \pm \sqrt{b^2 - 4ac}}{2a}.$$
Single function letters are also in italics, like $f(x)$ and e^x, not f(x) or e^{x} (yuck!). The Fibonacci numbers are written as F_n, not as F_n or (worse) F_{n}.
- Numbers are *never* in italics: a polynomial is $x^2 - 3x + 1$, not $x^2 - \mathit{3}x + \mathit{1}$. Basically, italic numbers look awful and should be avoided in all circumstances.
- Traditional functions whose label uses several letters are written in non-italic font: $\sin\theta$, $\cos\alpha$, and $\log t$, not $sin\ \theta$, $cos\ \alpha$, or $log\ t$.
- The statement of a lemma, theorem, or corollary is typeset in italics. A definition or example is not in italics, except for the term being defined. See the previous section for illustrations of this.

You should open up a math book and notice this traditional way of typesetting if you were not explicitly aware of it before.