Sec 7.2 Transformation of IVPs

Thm (Laplace transforms of derivatives)

[$f(t)$ is differentiable except possibly at finitely many points and $f'(t)$ is piecewise continuous]

Let $f(t)$ be a continuous and piecewise smooth function for $t \geq 0$

and $\underbrace{|f(t)| \leq Me^{ct} \text{ for some nonnegative constants } c, M.}_{f(t) \text{ is of exponential order as } t \to \infty}$

THEN

a) $\mathcal{L}\{f'(t)\}$ exists for $s > c$ and

b) $\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0)$.

Computation for part (b):

$$\mathcal{L}\{f'(t)\} \overset{\text{def}}{=} \lim_{N \to \infty} \int_0^N e^{-st} f'(t)\, dt$$

$u = e^{-st}$ $\quad dv = f'(t)\, dt$

$du = -se^{-st}$ $\quad v = f(t)$

$$= \lim_{N \to \infty} \left(e^{-st} f(t) \Big|_0^N - \int_0^N -se^{-st} f(t)\, dt \right)$$

$$= \lim_{N \to \infty} e^{-sN} f(N) - f(0) + s \lim_{N \to \infty} \int_0^N e^{-st} f(t)\, dt$$

$$= s\mathcal{L}\{f(t)\} - f(0)$$

From $\boxed{\mathcal{L}\{f'(t)\} = s\,\mathcal{L}\{f(t)\} - f(0)}$ (eq. 1)

we get

$$\mathcal{L}\{f''(t)\} = s\,\mathcal{L}\{f'(t)\} - f'(0)$$

$$= s\left(s\,\mathcal{L}\{f(t)\} - f(0)\right) - f'(0)$$

$$= s^2\,\mathcal{L}\{f(t)\} - s\,f(0) - f'(0)$$

$$\boxed{\mathcal{L}\{f''(t)\} = s^2\,\mathcal{L}\{f(t)\} - s\,f(0) - f'(0)} \quad \text{(eq. 2)}$$

We can continue with 3rd, 4th, ..., n-th derivative:

$$\mathcal{L}\{f^{(n)}(t)\} = s^n\,\mathcal{L}\{f(t)\} - s^{n-1}f(0) - s^{n-2}f'(0) - \dots - f^{(n-1)}(0)$$

Ex: Previously we computed

$$\mathcal{L}\{1\} = \int_0^\infty e^{-st}\,(1)\,dt$$

$$= \lim_{N\to\infty} \int_0^N e^{-st}\,dt$$

$$\int_0^N e^{-st}\,dt = \begin{cases} \int_0^N 1\,dt = N & \text{if } s=0 \\ -\frac{1}{s}e^{-st}\Big|_0^N & \text{if } s\neq 0 \end{cases}$$

$$= \frac{1}{s} \quad \text{if } s>0$$

Ex: Let $f(t) = t$.

(eq. 1) tells us $\mathcal{L}\{f'(t)\} = s\,\mathcal{L}\{f(t)\} - f(0)$,

so $\mathcal{L}\{1\} = s\,\mathcal{L}\{t\} - 0$

$$\frac{1}{s} = s\,\mathcal{L}\{t\}$$

$$\boxed{\mathcal{L}\{t\} = \frac{1}{s^2}} \quad \text{for } s>0$$

Ex: Compute $\mathcal{L}\{t^2\}$ using the fact that $\mathcal{L}\{1\} = \frac{1}{s}$ for $s > 0$.

Ans: Let $f(t) = t^2$

$f'(t) = 2t$

$f''(t) = 2 = 2.1$

(eq. 2) tells us that $\mathcal{L}\{f''(t)\} = s^2 \mathcal{L}\{f(t)\} - s f(0) - f'(0)$

So $\mathcal{L}\{2.1\} = s^2 \mathcal{L}\{t^2\} - s(0) - 0$

$$2\mathcal{L}\{1\} = s^2 \mathcal{L}\{t^2\}$$

$$2 \frac{1}{s} = s^2 \mathcal{L}\{t^2\}$$

$$\mathcal{L}\{t^2\} = \frac{2}{s^3} \quad \text{for } s > 0.$$

Ex: Solve the IVP $y'' - y' - 6y = \underset{\uparrow \text{ homogeneous}}{0}$, $y(0) = 2$, $y'(0) = -1$.

Ans:

Step 1 Take Laplace transform of both sides:

$$\mathcal{L}\{y'' - y' - 6y\} = \mathcal{L}\{0\}$$

Linearity of Laplace transform

$$\underbrace{\mathcal{L}\{y''\}} - \underbrace{\mathcal{L}\{y'\}} - 6\mathcal{L}\{y\} = 0$$

$$\left(s^2 \mathcal{L}\{y(t)\} - s\,y(0) - y'(0)\right) - \left(s\mathcal{L}\{y(t)\} - y(0)\right) - 6\mathcal{L}\{y\} = 0$$

$$(s^2 - s - 6)\mathcal{L}\{y(t)\} + (-s+1)\,y(0) - y'(0) = 0$$

$$(s^2 - s - 6)\mathcal{L}\{y(t)\} + (-s+1)\,2 + 1 = 0$$

$$(s^2 - s - 6)\mathcal{L}\{y(t)\} - 2s + 3 = 0$$

$$\mathcal{L}\{y(t)\} = \frac{2s-3}{s^2 - s - 6}$$

Next, take inverse Laplace transform, but first ...

Step 2 Write $\mathcal{L}\{y(t)\}$ as a sum of partial fraction

Denominator has distinct real roots 3 and -2

$$\mathcal{L}\{y(t)\} = \frac{2s-3}{s^2-s-6} = \frac{2s-3}{(s-3)(s+2)} = \frac{A}{s-3} + \frac{B}{s+2}$$

Multiply both sides by $(s-3)(s+2)$:

$$2s-3 = A(s+2) + B(s-3)$$

$$2s-3 = (A+B)s + (2A-3B)$$

$$\left.\begin{aligned} A+B &= 2 \\ 2A-3B &= -3 \end{aligned}\right\} \Rightarrow \quad \begin{aligned} &A = 2-B \\ &2(2-B) - 3B = -3 \Rightarrow 4-5B = -3 \Rightarrow 5B = 7 \end{aligned}$$

$$B = \frac{7}{5}, \quad A = 2 - \frac{7}{5} = \frac{3}{5}$$

(Alternatively, evaluate at the roots -2 and 3:

Set $s=-2$: $2(-2)-3 = A(0) + B(-5) \Rightarrow -7 = -5B \Rightarrow B = \frac{7}{5}$

Set $s=3$: $2(3)-3 = A(5) + B(0) \Rightarrow 3 = 5A \Rightarrow A = \frac{3}{5}$

This *only* works if the roots are all real and distinct.)

$$\mathcal{L}\{y(t)\} = \frac{2s-3}{s^2-s-6} = \frac{3}{5}\frac{1}{s-3} + \frac{7}{5}\frac{1}{s+2}$$

double check

Step 3 Find $y(t)$ by computing the inverse transform of the sum of partial fractions

Recall $\mathcal{L}\{e^{at}\} = \frac{1}{s-a}$

$$y(t) = \mathcal{L}^{-1}\left\{\frac{3}{5}\frac{1}{s-3} + \frac{7}{5}\frac{1}{s+2}\right\}$$

$$= \frac{3}{5}\mathcal{L}^{-1}\left\{\frac{1}{s-3}\right\} + \frac{7}{5}\mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\}$$

by linearity of inverse Laplace transform

$$= \boxed{\frac{3}{5}e^{3t} + \frac{7}{5}e^{-2t}} \quad \text{for } t \geq 0$$

Check:

- Double check that $y(t) = \frac{3}{5}e^{3t} + \frac{7}{5}e^{-2t}$ satisfies the IVP
- Use Chapter 3 method to arrive at the same solution.

Recall other possible situations for partial fractions
(See Recommended textbook problems for more computation examples)

* Denominator has non-real roots $\pm i$, $\pm 2i$
(we don't need to compute these roots)

Degree 1 polynomials

$$\frac{2}{(s^2+1)(s^2+4)} = \frac{As+B}{s^2+1} + \frac{Cs+D}{s^2+4}$$

See Sec 7.2 Problem 5 in textbook

* Denominator has non-real roots $\pm i$, $\pm 3i$

Degree 1 polynomials

$$\frac{s^3+10s}{(s^2+1)(s^2+9)} = \frac{As+B}{s^2+1} + \frac{Cs+D}{s^2+9}$$

See Sec 7.2 Problem 7 in textbook

* Denominator has a repeated root:

Degree 0 polynomials because $0, -1, -2$ are real roots

$$\frac{1+2s^2}{s^2(s+1)(s+2)} = \underbrace{\frac{A}{s} + \frac{B}{s^2}}_{} + \frac{C}{s+1} + \frac{D}{s+2}$$

because root 0 is repeated twice
(multiplicity of root 0 is 2)

See Sec 7.2 Problem 10 in textbook

Ex: Use Laplace transforms to solve the IVP

$$y'' + 4y = \sin(3t), \quad y(0) = 0, \quad y'(0) = 0$$

non-homogeneous

Ans: Step 1 Take $\mathcal{L}$ of both sides to get $\mathcal{L}\{y(t)\}$:

- $\mathcal{L}\{y''+4y\} = \mathcal{L}\{\sin(3t)\}$
- $\mathcal{L}\{y''\} = s^2\mathcal{L}\{y(t)\} - s\,y(0) - y'(0)$ from (eq. 2)
  $= s^2\mathcal{L}\{y(t)\}$
- $s^2\mathcal{L}\{y(t)\} + 4\mathcal{L}\{y(t)\} = \dfrac{3}{s^2+9}$ from Sec 7.1 table

$$(s^2+4)\,\mathcal{L}\{y(t)\} = \frac{3}{s^2+9}$$

$$\mathcal{L}\{y(t)\} = \frac{3}{(s^2+9)(s^2+4)}$$

Step 2 Write $\mathcal{L}\{y(t)\}$ as a sum of partial fraction

- Roots of the denominator are $\pm 3i$, $\pm 2i$, each with multiplicity 1.

Degree 1 polynomials

$$\mathcal{L}\{y(t)\} = \frac{3}{(s^2+9)(s^2+4)} = \frac{As+B}{s^2+9} + \frac{Cs+D}{s^2+4}$$

- Multiply both sides by the denominator

$$3 = (As+B)(s^2+4) + (Cs+D)(s^2+9)$$

- Group by s^3, s^2, s, 1 :

$$3 = As^3 + 4As + Bs^2 + 4B + Cs^3 + 9Cs + Ds^2 + 9D$$

$$3 = \underbrace{(A+C)}_{=0}s^3 + \underbrace{(B+D)}_{=0}s^2 + \underbrace{(4A+9C)}_{=0}s + \underbrace{(4B+9D)}_{=3}$$

$$\left.\begin{array}{llll} A & & +C & = 0 \\ & B & & +D = 0 \\ 4A & & +9C & = 0 \\ & 4B & & +9D = 3 \end{array}\right\} \Rightarrow A = -C$$

$$4(-C) + 9C = 0 \Rightarrow 5C = 0 \Rightarrow \boxed{C=0} \quad \boxed{A=0}$$

$$B = -D$$

$$4(-D) + 9D = 3 \Rightarrow 5D = 3 \Rightarrow \boxed{D = \tfrac{3}{5}} \Rightarrow \boxed{B = -\tfrac{3}{5}}$$

- $$\mathcal{L}\{y(t)\} = \frac{3}{(s^2+9)(s^2+4)} = -\frac{3}{5}\frac{1}{s^2+9} + \frac{3}{5}\frac{1}{s^2+4}$$

Step 3 Find $y(t)$ by computing the inverse transform of the sum of partial fractions

- From Sec 7.1 table, $\mathcal{L}\{\sin(at)\} = \dfrac{a}{s^2+a^2}$

- $$y(t) = \mathcal{L}^{-1}\left\{-\frac{3}{5}\frac{1}{s^2+9} + \frac{3}{5}\frac{1}{s^2+4}\right\}$$

$$= -\frac{1}{5}\mathcal{L}^{-1}\left\{\frac{3}{s^2+9}\right\} + \frac{3}{5}\cdot\frac{1}{2}\mathcal{L}^{-1}\left\{\frac{2}{s^2+4}\right\}$$

Make sure to put "a" in numerator

$$= \boxed{-\frac{1}{5}\sin(3t) + \frac{3}{10}\sin(2t)} \quad \text{for } t \geq 0$$

Check:

- Double check that $y(t)$ satisfies the IVP
- Use Chapter 3 method to arrive at the same solution. ▨

Ex:

Use the formula $\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0)$ from the "Laplace transforms of derivatives" thm to find the Laplace transform of $f(t) = te^{7t}$

Ans: $f'(t) = e^{7t} + 7t\,e^{7t}$

$$\mathcal{L}\{e^{7t} + 7t\,e^{7t}\} = s\,\mathcal{L}\{te^{7t}\} - 0\,e^{7\cdot 0}$$

$$\mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} - f(0)$$

$$\mathcal{L}\{e^{7t}\} + 7\mathcal{L}\{te^{7t}\} = s\,\mathcal{L}\{te^{7t}\} - 0$$

$$\mathcal{L}\{e^{7t}\} = (s-7)\,\mathcal{L}\{te^{7t}\}$$

equal for $s > 7$

$$\frac{1}{s-7} = (s-7)\,\mathcal{L}\{te^{7t}\}$$

$$\boxed{\mathcal{L}\{te^{7t}\} = \frac{1}{(s-7)^2} \quad \text{for} \quad s > 7}$$

In general ...

Example 4 Show that

$$\mathcal{L}\{te^{at}\} = \frac{1}{(s-a)^2}.$$

Solution If $f(t) = te^{at}$, then $f(0) = 0$ and $f'(t) = e^{at} + ate^{at}$. Hence Theorem 1 gives

$$\mathcal{L}\{e^{at} + ate^{at}\} = \mathcal{L}\{f'(t)\} = s\mathcal{L}\{f(t)\} = s\mathcal{L}\{te^{at}\}.$$

It follows from the linearity of the transform that

$$\mathcal{L}\{e^{at}\} + a\mathcal{L}\{te^{at}\} = s\mathcal{L}\{te^{at}\}.$$

Hence

$$\mathcal{L}\{te^{at}\} = \frac{\mathcal{L}\{e^{at}\}}{s-a} = \frac{1}{(s-a)^2}$$

because $\mathcal{L}\{e^{at}\} = 1/(s-a)$.

THEOREM 2 Transforms of Integrals

If $f(t)$ is a piecewise continuous function for $t \geqq 0$ and satisfies the condition of exponential order $|f(t)| \leqq Me^{ct}$ for $t \geqq T$, then

$$\mathcal{L}\left\{\int_0^t f(\tau)\,d\tau\right\} = \frac{1}{s}\mathcal{L}\{f(t)\} = \frac{F(s)}{s} \tag{17}$$

for $s > c$. Equivalently,

$$\mathcal{L}^{-1}\left\{\frac{F(s)}{s}\right\} = \int_0^t f(\tau)\,d\tau. \tag{18}$$

Ex:

$$\mathcal{L}^{-1}\left\{\frac{1}{s(s-a)}\right\} = \int_0^t \mathcal{L}^{-1}\left\{\frac{1}{s-a}\right\}d\tau = \int_0^t e^{a\tau}\,d\tau = \frac{1}{a}(e^{at}-1).$$

Ex:

$$\mathcal{L}^{-1}\left\{\frac{1}{s^2(s-a)}\right\} = \int_0^t \mathcal{L}^{-1}\left\{\frac{1}{s(s-a)}\right\}d\tau = \int_0^t \frac{1}{a}(e^{a\tau}-1)\,d\tau$$

$$= \left[\frac{1}{a}\left(\frac{1}{a}e^{a\tau} - \tau\right)\right]_0^t = \frac{1}{a^2}(e^{at} - at - 1).$$