7.1 Laplace transforms and inverse transforms

Motivation: Consider an ODE

$$m y'' + c y' + k y = F(t)$$ (corresponding to "vibrating spring-mass" system)

This is a 2nd-order linear ODE.

If $F(t)$ is continuous, then if we specify initial conditions we know the solution exists and is unique.

Q: What if $F(t)$ is not continuous?

A class of functions which are almost as nice as continuous functions is the class of <u>piecewise continuous</u> functions.

Ex:

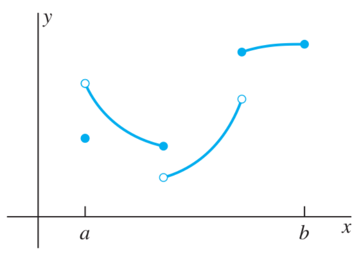


FIGURE 7.1.3. The graph of a piecewise continuous function; the solid dots indicate values of the function at discontinuities.

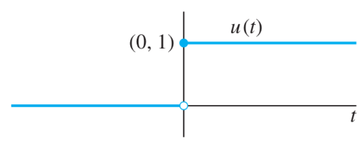


FIGURE 7.1.4. The graph of the unit step function.

To deal with these situation, we'll use Laplace transforms.

$$\left\{\begin{matrix}\text{differential}\\ \text{equations}\end{matrix}\right\} \xrightarrow{\text{Laplace transform}} \left\{\begin{matrix}\text{algebraic}\\ \text{equations}\end{matrix}\right\}$$

containing derivatives

only algebraic operations: addition, multiplication, raising to a power, taking root

Recall <u>Definition of improper integrals</u>

Note: An <u>improper integral</u> is <u>not</u> an integral

* Let a be a real number and consider the definite integral $\int_a^b g(t)\, dt$ if $b > a$.

* If $\lim_{N\to\infty} \int_a^N g(t)\, dt$ exists

(that is, $\int_a^N g(t)\, dt$ goes to a number as $N \to \infty$)

then we write

$$\int_a^\infty g(t)\, dt \overset{\text{def}}{:=} \lim_{N\to\infty} \int_a^N g(t)\, dt \quad \text{and}$$

This is an example of an <u>improper integral</u>

we say $\int_a^\infty g(t)\, dt$ <u>converges</u>.

* If the limit doesn't exist, we say $\int_a^\infty g(t)\, dt$ <u>diverges</u>.

(possibly $\int_a^\infty g(t)\, dt = \infty$)

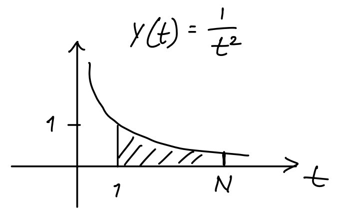
EX: Evaluate the improper integral $\int_1^\infty \frac{1}{t^2}\,dt$.

Ans

* If $N > 1$, then $\int_1^N \frac{1}{t^2}\,dt = \int_1^N t^{-2}\,dt$

$$= -t^{-1} \Big|_{t=1}^{t=N}$$

$$= -\frac{1}{N} + 1$$



* So $\int_1^\infty \frac{1}{t^2}\,dt = \lim_{N\to\infty} \int_1^N \frac{1}{t^2}\,dt$

$$= \lim_{N\to\infty} -\frac{1}{N} + 1$$

$$= 1$$

We say $\int_1^\infty \frac{1}{t^2}\,dt$ converges

Definition of Laplace transform

* Let $f(t)$ be a function defined for all $t \geq 0$.

The Laplace transform of $f(t)$, denoted by $\mathcal{L}\{f(t)\}$,

is a function of s defined as

$$F(s) = \mathcal{L}\{f(t)\} \stackrel{\text{def}}{=} \int_0^\infty e^{-st} f(t)\,dt$$

for all numbers s for which the improper integral converges

This says that the domain of the function $F(s)$

is the set of numbers s for which $\int_0^\infty e^{-st} f(t)\,dt$ converges

Review Integration by parts!

Ex: Use the definition to find the Laplace transform $\mathcal{L}\{f(t)\}$ of the function $f(t) = e^{3t}$ and the domain of $\mathcal{L}\{f(t)\}$.

<u>Ans</u> * If $N > 0$, then $\displaystyle\int_0^N e^{-st} e^{3t}\, dt = \int_0^N e^{(-s+3)t}\, dt$

$$= \left. \frac{e^{(-s+3)t}}{-s+3} \right|_{t=0}^{t=N}$$

$$= \frac{1}{-s+3}\left[e^{(-s+3)N} - 1\right]$$

* $\displaystyle F(s) = \mathcal{L}\{e^{3t}\} := \int_0^\infty e^{-st} e^{3t}\, dt$

$$= \lim_{N\to\infty} \int_0^N e^{-st} e^{3t}\, dt$$

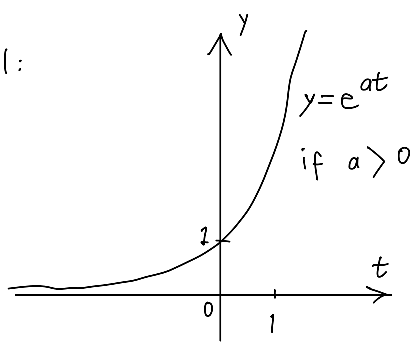
$$= \lim_{N\to\infty} \frac{1}{-s+3}\left[e^{(-s+3)N} - 1\right]$$

$$= \frac{1}{s-3} \quad \text{if } (-s+3) < 0$$ (see below)

$$s > 3$$

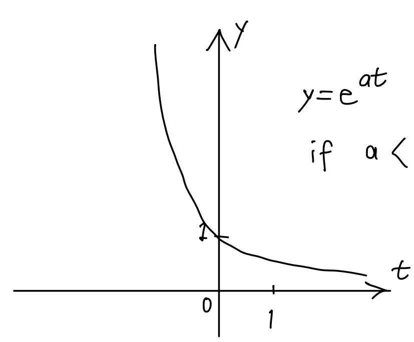
* So $F(s) = \mathcal{L}\{e^{3t}\} = \dfrac{1}{s-3}$ and the domain of $F(s)$ is $(3, \infty)$

Recall:



If $a > 0$, $\displaystyle\lim_{N\to\infty} e^{aN} = \infty$

so $\displaystyle\lim_{N\to\infty} e^{aN}$ doesn't exist



If $a < 0$, $\displaystyle\lim_{N\to\infty} e^{aN} = 0$

Q: What is the Laplace transform of e^{7t} ?

Ans: $\mathcal{L}\{e^{7t}\} = \frac{1}{s-7}$ for $s > 7$

domain is $(7, \infty)$

Q: What is the Laplace transform of e^{-t} ?

Ans: $\mathcal{L}\{e^{-t}\} = \frac{1}{s+1}$ for $s > -1$

domain is $(-1, \infty)$

Q: What is the Laplace transform of e^{at} ?

Ans: $\mathcal{L}\{e^{at}\} = \frac{1}{s-a}$ for $s > a$

domain is (a, ∞)

Thm 1 (Linearity of the Laplace transform)

Let $c_1, c_2 \in \mathbb{R}$. Then

$$\mathcal{L}\{c_1 f(t) + c_2 g(t)\} = c_1 \mathcal{L}\{f(t)\} + c_2 \mathcal{L}\{g(t)\}$$

for all s such that both $\mathcal{L}\{f(t)\}$ and $\mathcal{L}\{g(t)\}$ exist.

(The linearity of Laplace transform comes from the linearity of integration.)

Ex: $\mathcal{L}\{2e^{3t} - 100e^{7t} + e^{-t}\} = 2\mathcal{L}\{e^{3t}\} - 100\mathcal{L}\{e^{7t}\} + \mathcal{L}\{e^{-t}\}$

$$= 2\frac{1}{s-3} - 100\frac{1}{s-7} + \frac{1}{s+1}$$

for $s > 7$.

(The domain is $(7, \infty)$ because it is the intersection of $(3, \infty)$, $(7, \infty)$, and $(-1, \infty)$.)

Thm 2 (Existence of Laplace transform)

IF:

a.) $f(t)$ is a piecewise continuous function for $t \geqslant 0$

b.) $\underbrace{|f(t)| \leqslant M e^{ct} \text{ for some constants } M, c}$

say "$f(t)$ is of exponential order as $t \to \infty$"

THEN:

the Laplace transform $F(s) = \mathcal{L}\{f(t)\}$ exists for all $s > c$.

Say:

"A piecewise continuous function with controlled exponential growth has a Laplace transform."

Corollary ← meaning a consequence of this thm

If $f(t)$ satisfies conditions (a) and (b) from the above thm,

then $\lim\limits_{s \to \infty} F(s) = 0$

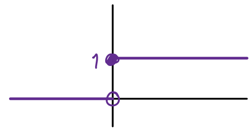
This means that only functions $F(s)$ such that $\lim\limits_{s\to\infty} F(s) = 0$ can be Laplace transforms of "reasonable" functions.

Ex $G(s) = \dfrac{s}{s+1}$ cannot be the Laplace transform of a resonable function because $\lim\limits_{s\to\infty} G(s) = 1 \neq 0$.

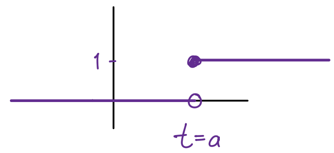
Examples of piecewise continuous functions

Unit step functions (Ex 1 & 2)

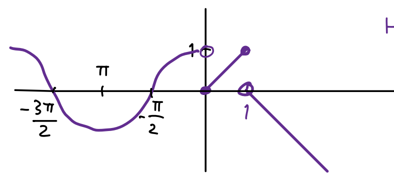
Ex 1. $u(t) = \begin{cases} 0 & \text{for } t < 0 \\ 1 & \text{for } t \geqslant 0 \end{cases}$



Ex 2. $u_a(t) = u(t-a) = \begin{cases} 0 & \text{for } t - a < 0 \\ & (\text{for } t < a) \\ 1 & \text{for } t - a \geqslant 0 \\ & (\text{for } t \geqslant a) \end{cases}$



Ex 3. $f(t) = \begin{cases} \cos(t) & \text{for } t < 0 \\ t & \text{for } 0 \leq t \leq 1 \\ -t+1 & \text{for } t > 1 \end{cases}$



Here $\lim\limits_{t \to 0^-} f(t) = 1$

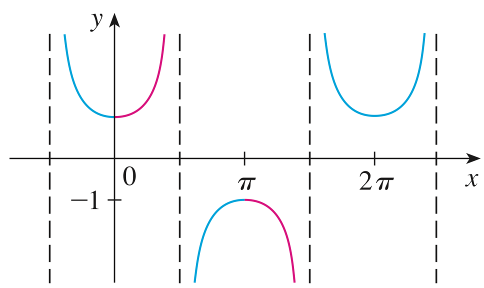
$\lim\limits_{t \to 0^+} f(t) = 0$

Ex 4. Not a piecewise continuous function

$f(t) = \dfrac{1}{\cos t}$

because, e.g., $\lim\limits_{x \to \left(\frac{\pi}{2}\right)^-} f(t) = \infty$

limit doesn't exist



Ex 1: $\mathcal{L}\{u(t)\} = \int_0^\infty e^{-st} u(t)\, dt = \int_0^\infty e^{-st}\, dt = \lim_{N\to\infty} \left. \frac{e^{-st}}{-s} \right|_{t=0}^{t=N}$

$= \lim_{N\to\infty} -\frac{1}{s}\left(e^{-sN} - 1\right)$

$= \frac{1}{s}$ for $s > 0$

(domain is $(0,\infty)$)

Ex 2: If $a > 0$,

$\mathcal{L}\{u_a(t)\} = \int_0^\infty e^{-st} u_a(t)\, dt = \int_0^a e^{-st} u_a(t)\, dt + \int_a^\infty e^{-st} u_a(t)\, dt$

$= \int_a^\infty e^{-st}\, dt = \lim_{N\to\infty} -\frac{1}{s}\left(e^{-sN} - e^{-sa}\right) = \frac{e^{-as}}{s}$ for $s > 0$

(domain is $(0,\infty)$)

Ex 3: $\mathcal{L}\{f(t)\} = \int_0^\infty e^{-st} f(t)\, dt = \int_0^1 e^{-st}\, t\, dt + \int_1^\infty e^{-st} (-t+1)\, dt$

(from Ex 3 above)

definite integral

Improper integral

Finish using integration by parts

$\boxed{\text{The gamma function}}$ $\Gamma(x) = \int_0^\infty e^{-t} t^{x-1}\, dt$ defined for $x > 0$

Some properties:

$\Gamma(1) = 1$ and $\Gamma(x+1) = x\Gamma(x)$ for $x > 0$

Ex: $\Gamma(5) = \Gamma(4+1)$

$= 4\Gamma(4)$

$= 4.3\Gamma(3)$

$= 4.3.2\Gamma(2)$

$= 4.3.2.1.\Gamma(1)$

$= 4!$

$n! = n(n-1)(n-2)\ldots 3.2.1$

$\Gamma(n+1) = n!$

$$\Gamma\left(\tfrac{1}{2}\right) = \sqrt{\pi}$$

$$\begin{aligned}
\Gamma\left(\tfrac{5}{2}\right) &= \Gamma\left(\tfrac{3}{2}+1\right) \\
&= \tfrac{3}{2}\,\Gamma\left(\tfrac{3}{2}\right) \\
&= \frac{3}{2}\,\Gamma\left(\tfrac{1}{2}+1\right) \\
&= \frac{3}{2}\,\frac{1}{2}\,\Gamma\left(\tfrac{1}{2}\right) \\
&= \frac{3}{4}\sqrt{\pi}
\end{aligned}$$

The gamma function is useful for computing the Laplace transforms of powers of t: Let $f(t) = t^a$ where $a \in \mathbb{R}$ and $a > -1$.

$$\mathcal{L}\{t^a\} = \int_0^\infty e^{-st}\, t^a \, dt \qquad \left(\begin{array}{ll} u = st & \\ \frac{u}{s} = t & dt = \frac{1}{s}\,du \end{array} \right)$$

$$\begin{aligned}
&= \int_{u=0}^{u=\infty} e^{-u} \left(\frac{u}{s}\right)^a \frac{1}{s}\,du \\
&= \frac{1}{s^{a+1}} \int_0^\infty e^{-u}\, u^a \, du \\
&= \frac{1}{s^{a+1}}\,\Gamma(a+1) \qquad \text{for all } s > 0
\end{aligned}$$

Then

$$\left.\begin{aligned}
\mathcal{L}\{t\} &= \frac{1}{s^2}\,\Gamma(2) = \frac{1}{s^2}\,1! = \frac{1}{s^2} \\
\mathcal{L}\{t^2\} &= \frac{1}{s^3}\,\Gamma(3) = \frac{1}{s^3}\,2! = \frac{2}{s^3} \\
\mathcal{L}\{t^3\} &= \frac{1}{s^4}\,\Gamma(4) = \frac{1}{s^4}\,3! = \frac{6}{s^4} \\
\mathcal{L}\{t^n\} &= \frac{n!}{s^{n+1}}
\end{aligned}\right\} \quad \text{for } s > 0$$

By linearity of Laplace transforms, we have

$$\mathcal{L}\{3t^2 + 4t^{\frac{3}{2}} - 7\} = 3\,\mathcal{L}\{t^2\} + 4\,\mathcal{L}\{t^{\frac{3}{2}}\} - 7\,\mathcal{L}\{1\}$$

$1 = t^0$

$f(t)$	$F(s)$	
1	$\frac{1}{s}$	$(s > 0)$
t	$\frac{1}{s^2}$	$(s > 0)$
$t^n \quad (n \geqq 0)$	$\frac{n!}{s^{n+1}}$	$(s > 0)$
$t^a \quad (a > -1)$	$\frac{\Gamma(a+1)}{s^{a+1}}$	$(s > 0)$
e^{at}	$\frac{1}{s-a}$	$(s > a)$
$\cos kt$	$\frac{s}{s^2+k^2}$	$(s > 0)$
$\sin kt$	$\frac{k}{s^2+k^2}$	$(s > 0)$
$\cosh kt$	$\frac{s}{s^2-k^2}$	$(s > \lvert k \rvert)$
$\sinh kt$	$\frac{k}{s^2-k^2}$	$(s > \lvert k \rvert)$
$u(t-a)$	$\frac{e^{-as}}{s}$	$(s > 0)$

FIGURE 7.1.2. A short table of Laplace transforms.

$$= 3\,\frac{2}{s^3} + 4\,\frac{1}{s^{\frac{5}{2}}}\,\Gamma\left(\tfrac{5}{2}\right) - 7\,\frac{1}{s}$$

$$= \frac{6}{s^3} + 4\,\frac{1}{s^{5/2}}\,\underbrace{\frac{3}{4}\sqrt{\pi}}_{\text{from earlier}} - \frac{7}{s}$$

$$= \frac{6}{s^3} + \frac{3\sqrt{\pi}}{s^{5/2}} - \frac{7}{s}$$

Inverse Laplace transforms

Def: If $F(s) = \mathcal{L}\{f(t)\}$ is the Laplace transform of $f(t)$, then we say that

$$f(t) = \mathcal{L}^{-1}\{F(s)\}$$

is the <u>inverse Laplace transform</u> of $F(s)$.

Ex 1: $\mathcal{L}^{-1}\left\{\frac{1}{s}\right\} = 1$ because $\mathcal{L}\{1\} = \frac{1}{s}$

domain of $F(s) = \frac{1}{s}$ is $(0, \infty)$

domain of $f(t) = 1$ is $[0, \infty)$

Ex 2: $\mathcal{L}^{-1}\left\{\frac{1}{s+2}\right\} = e^{-2t}$ because $\mathcal{L}\{e^{-2t}\} = \frac{1}{s+2}$

domain of $F(s) = \frac{1}{s+2}$ is $(-2, \infty)$

domain of $f(t) = e^{-2t}$ is $[0, \infty)$

Linearity of inverse transform

Because of the linearity of the Laplace transform, the inverse transform is also linear.

Ex 1 $\quad \mathcal{L}^{-1}\left\{\frac{1}{s^3}\right\} = \mathcal{L}^{-1}\left\{\frac{1}{2}\, 2\, \frac{1}{s^3}\right\} = \frac{1}{2}\mathcal{L}^{-1}\left\{\frac{2}{s^3}\right\} = \frac{1}{2} t^2$

domain of $F(s)$ is $(0, \infty)$

linearity of $\mathcal{L}^{-1}$

from the table (fig 7.1.2)

domain of $f(t)$ is $[0, \infty)$

linearity of $\mathcal{L}^{-1}$

Ex 2 $\quad \mathcal{L}^{-1}\left\{\underbrace{\frac{3}{s-2} + \frac{1}{s} - \frac{s}{s^2+36}}_{\text{domain of } F(s) \text{ is } (2,\infty)}\right\} = 3\mathcal{L}^{-1}\left\{\frac{1}{s-2}\right\} + \mathcal{L}^{-1}\left\{\frac{1}{s}\right\} - \mathcal{L}^{-1}\left\{\frac{s}{s^2+36}\right\}$

from table $\quad = \underbrace{3e^{2t} + 1 - \cos(6t)}_{\text{domain of } f(t) \text{ is } [0,\infty)}$

$f(t)$	$F(s)$	
1	$\dfrac{1}{s}$	$(s > 0)$
t	$\dfrac{1}{s^2}$	$(s > 0)$
$t^n \quad (n \geqq 0)$	$\dfrac{n!}{s^{n+1}}$	$(s > 0)$
$t^a \quad (a > -1)$	$\dfrac{\Gamma(a+1)}{s^{a+1}}$	$(s > 0)$
e^{at}	$\dfrac{1}{s-a}$	$(s > a)$
$\cos kt$	$\dfrac{s}{s^2+k^2}$	$(s > 0)$
$\sin kt$	$\dfrac{k}{s^2+k^2}$	$(s > 0)$
$\cosh kt$	$\dfrac{s}{s^2-k^2}$	$(s > \lvert k \rvert)$
$\sinh kt$	$\dfrac{k}{s^2-k^2}$	$(s > \lvert k \rvert)$
$u(t-a)$	$\dfrac{e^{-as}}{s}$	$(s > 0)$

FIGURE 7.1.2. A short table of Laplace transforms.

Thm 3 (Uniqueness of inverse Laplace transform)

Suppose $f(t)$ and $g(t)$ are such that

$F(s) = \mathcal{L}\{f(t)\}$ and $G(s) = \mathcal{L}\{g(t)\}$.

If $F(s) = G(s)$ for all $s > c$ for some $c \in \mathbb{R}$,

then $f(t) = g(t)$ wherever on $[0, \infty)$ both f and g are continuous.