A little bit of linear algebra for Sec 5.1, 5.2

- A <u>matrix</u> is a block of numbers, ex: $M = \begin{pmatrix} 1 & -2 & 3 \\ 4 & 5 & \frac{1}{6} \end{pmatrix}$

- The size of a matrix M is $m \times n$ if M has m rows and n columns

  size of M is 2×3

- Read rows from top to bottom
  Read columns from left to right

  Row 1
  Row 2
  $$\begin{pmatrix} 1 & -2 & 3 \\ 4 & 5 & \frac{1}{6} \end{pmatrix}$$

  $$\begin{array}{c} \begin{pmatrix} 1 & -2 & 3 \\ 4 & 5 & \frac{1}{6} \end{pmatrix} \\ \text{col 1} \quad \text{col 2} \quad \text{col 3} \end{array}$$

- The (i,j)-th element of a matrix M is the entry on the i-th row and j-th column of M

  $$M = \begin{pmatrix} 1 & -2 & 3 \\ 4 & 5 & \frac{1}{6} \end{pmatrix}$$

  $(1,3)$-th element of M is 3

  $(2,1)$-th element of M is 4

- Matrix addition / subtraction:

  * If A, B are matrices of the same size, then we add/subtract the entries to get $A+B$ and $A-B$

    $$A = \begin{pmatrix} 2 & 1 & 4 \\ 3 & 0 & 6 \end{pmatrix}, \quad B = \begin{pmatrix} -1 & 3 & -2 \\ 1 & -2 & 1 \end{pmatrix}$$

    $$A+B = \begin{pmatrix} 2-1 & 1+3 & 4-2 \\ 3+1 & -2 & 6+1 \end{pmatrix} = \begin{pmatrix} 1 & 4 & 2 \\ 4 & -2 & 7 \end{pmatrix}$$

    Note: $A+B = B+A$ (matrix addition is commutative)

    $$A-B = \begin{pmatrix} 2--1 & 1-3 & 4--2 \\ 3-1 & --2 & 6-1 \end{pmatrix} = \begin{pmatrix} 3 & -2 & 6 \\ 2 & 2 & 5 \end{pmatrix}$$

  * If A, B have different sizes, then $A+B$ and $A-B$ are not defined

    $$\begin{pmatrix} 2 & 1 \\ -1 & 4 \end{pmatrix} + \begin{pmatrix} 3 & 1 & 4 \\ 2 & 6 & 8 \end{pmatrix} \text{ is not defined}$$

- $\underbrace{\text{Scalar}}_{\text{"number"}}$ multiplication:

  If M is a matrix and c is a number, then cM is the matrix obtained by multiplying c to every entry in M

  Ex: $M = \begin{pmatrix} 1 & -2 & 3 \\ 4 & 5 & \frac{1}{6} \end{pmatrix}$, $c = 100$

  $$cM = 100 \begin{pmatrix} 1 & -2 & 3 \\ 4 & 5 & \frac{1}{6} \end{pmatrix} = \begin{pmatrix} 100 & -200 & 300 \\ 400 & 500 & \frac{100}{6} \end{pmatrix}$$

• Matrix multiplication:

$A = \begin{pmatrix} 2 & 1 & -1 \\ 0 & 3 & 2 \end{pmatrix}$, $B = \begin{pmatrix} 1 & 2 \\ -1 & 0 \\ 3 & -1 \end{pmatrix}$

size 2×3 size 3×2

* Let A be an $m \times n$ matrix and let B be an $n \times k$ matrix.

Then AB is an $m \times k$ matrix.

The (i,j)-th element of AB is the "dot product" of the i-th row of A and the j-th column of B

$$AB = \begin{pmatrix} 2 & 1 & -1 \\ 0 & 3 & 2 \end{pmatrix} \begin{pmatrix} 1 & 2 \\ -1 & 0 \\ 3 & -1 \end{pmatrix} = \begin{pmatrix} 2(1)+1(-1)-1(3) & 2(2)+1(0)-1(-1) \\ 0(1)+3(-1)+2(3) & 0(2)+3(0)+2(-1) \end{pmatrix}$$

$$= \begin{pmatrix} -2 & 5 \\ 3 & -2 \end{pmatrix}$$

size of AB is 2×2

* Note in general $AB \neq BA$, ex:

(Matrix multiplication is <u>not</u> commutative)

$$BA = \underset{3\times 2}{\begin{pmatrix} 1 & 2 \\ -1 & 0 \\ 3 & -1 \end{pmatrix}} \underset{2\times 3}{\begin{pmatrix} 2 & 1 & -1 \\ 0 & 3 & 2 \end{pmatrix}} = \begin{pmatrix} 1(2)+2(0) & 1(1)+2(3) & 1(-1)+2(2) \\ -1(2)+0(0) & -1(1)+0(3) & -1(-1)+0(2) \\ 3(2)-1(0) & 3(1)-1(3) & 3(-1)-1(2) \end{pmatrix}$$

* If (# of columns of A) $\neq$ (# of rows of B),

then AB is not defined, ex: $\underset{2\times 3}{\begin{pmatrix} 2 & 1 & -1 \\ 0 & 3 & 2 \end{pmatrix}} \underset{2\times 3}{\begin{pmatrix} 2 & 1 & -1 \\ 0 & 3 & 2 \end{pmatrix}}$ is not defined

not equal

• Identity matrices

* The <u>identity</u> $n \times n$ matrix is $I_n = I_{n\times n} = \begin{pmatrix} 1 & 0 & 0 & \cdots & 0 \\ 0 & 1 & & & 0 \\ \vdots & & 1 & & \vdots \\ \vdots & & & \ddots & 0 \\ 0 & 0 & 0 & \cdots 0 & 1 \end{pmatrix}$

Ex: $I_2 = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$, $I_3 = \begin{pmatrix} 1 & 0 & 0 \\ 0 & 1 & 0 \\ 0 & 0 & 1 \end{pmatrix}$

* Note: If M is an $n \times n$ matrix, then $M I_n = I_n M = M$.

(So the identity matrix acts like the number 1.)

Ex: check $\begin{pmatrix} 2 & 1 \\ -1 & 4 \end{pmatrix}\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} \overset{?}{=} \cdots$ $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\begin{pmatrix} 2 & 1 \\ -1 & 4 \end{pmatrix} \overset{?}{=} \cdots$

· Invertibility

An $n \times n$ matrix M is called <u>invertible</u> if there exists a matrix B such that $MB = BM = I_n$.

The matrix B is called <u>the</u> <u>inverse</u> of M and denoted $B = M^{-1}$

(fact: unique)

Note: Non-square matrices don't have inverses.

Ex: $M = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$, $B = \begin{pmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{pmatrix}$

$$MB = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}\begin{pmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{pmatrix} = \begin{pmatrix} 1(-2) + 2\left(\frac{3}{2}\right) & 1(1) + 2\left(-\frac{1}{2}\right) \\ 3(-2) + 4\left(\frac{3}{2}\right) & 3(1) + 4\left(-\frac{1}{2}\right) \end{pmatrix} = \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$$

Compute $BM = \begin{pmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{pmatrix}\begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix} \overset{?}{=} \ldots$

So M is invertible and $M^{-1} = \begin{pmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{pmatrix}$

B is invertible and $B^{-1} = M$

<u>Thm</u>

Let $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$. Then:

① M is invertible iff $ad - bc \neq 0$

Note: If $ad - bc \neq 0$ then M is invertible

If $ad - bc = 0$ then M is not invertible

② If $ad - bc \neq 0$ then $M^{-1} = \dfrac{1}{ad-bc}\begin{pmatrix} d & -b \\ -c & a \end{pmatrix}$

Definition

The <u>determinant</u> of a 2×2 matrix $M = \begin{pmatrix} a & b \\ c & d \end{pmatrix}$ is $\underbrace{\det(M)}_{\text{or } |M|} = ad - bc$

Ex: $M = \begin{pmatrix} 1 & 2 \\ 3 & 4 \end{pmatrix}$

$\det(M) = 1(4) - 2(3) = -2$ which is non zero, so M is invertible.

$$M^{-1} = \frac{1}{-2}\begin{pmatrix} 4 & -2 \\ -3 & 1 \end{pmatrix} = \begin{pmatrix} -2 & 1 \\ \frac{3}{2} & -\frac{1}{2} \end{pmatrix}$$

Ex: $M = \begin{pmatrix} 1 & 2 \\ -3 & -6 \end{pmatrix}$

$\det(M) = 1(-6) - 2(-3) = 0$, so M is <u>not</u> invertible.

Definition

Let M be an $n \times n$ matrix.

A number λ (lambda) is an <u>eigenvalue</u> of M if there exists a <u>nonzero</u> $\underbrace{\text{vector}}_{n\times 1 \text{ matrix}}$ v such that $Av = \lambda v$.

Such nonzero vector v is called an <u>eigenvector</u> of M.

Ex: $M = \begin{pmatrix} 1 & 4 \\ 1 & 1 \end{pmatrix}$

Qa: Is $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ an eigenvector of M?

Answer: $M\begin{pmatrix} 2 \\ 1 \end{pmatrix} = \begin{pmatrix} 1(2) + 4(1) \\ 1(2) + 1(1) \end{pmatrix} = \begin{pmatrix} 6 \\ 3 \end{pmatrix} = \overset{\lambda}{3}\begin{pmatrix} 2 \\ 1 \end{pmatrix}$

Yes, $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ is an eigenvector of M. Its corresponding eigenvalue is $\lambda = 3$

Qb: Is $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ an eigenvector of M?

Answer: $M\begin{pmatrix} 2 \\ 0 \end{pmatrix} = \begin{pmatrix} 1(2) + 4(0) \\ 1(2) + 1(0) \end{pmatrix} = \begin{pmatrix} 2 \\ 2 \end{pmatrix}$

If $M\begin{pmatrix} 2 \\ 0 \end{pmatrix} = \lambda\begin{pmatrix} 2 \\ 0 \end{pmatrix}$, then $\begin{pmatrix} 2 \\ 2 \end{pmatrix} = \lambda\begin{pmatrix} 2 \\ 0 \end{pmatrix}$ so

$2 = \lambda 2 \Rightarrow \lambda = 1$

$2 = \lambda 0 \Rightarrow 2 = (1)0$, impossible

So, <u>no</u>, $\begin{pmatrix} 1 \\ 0 \end{pmatrix}$ is not an eigenvector of M.

- How do we find all eigenvalues of a matrix?

Ex: Find all eigenvalues of $M = \begin{pmatrix} 1 & 4 \\ 1 & 1 \end{pmatrix}$

Ans: * We want to find a number λ and a nonzero 2×1 vector v such that $Mv = \lambda v$.

* We can subtract $\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix} v$ from both sides to get

$$\left(M - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right) v = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

* Fact: $Av = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ has a solution other than $v = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$ iff $\det(A) = 0$

Therefore λ is an eigenvalue and v is an eigenvector iff $\det\left(M - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right) = 0$

* Set $\det\left(\begin{bmatrix} 1-\lambda & 4-0 \\ 1-0 & 1-\lambda \end{bmatrix}\right) = 0$ (You can go directly to this step)

$$(1-\lambda)(1-\lambda) - (1)(4) = 0$$

$$1 - 2\lambda + \lambda^2 - 4 = 0$$

$$\lambda^2 - 2\lambda - 3 = 0$$

$$(\lambda + 1)(\lambda - 3) = 0$$

$\boxed{\lambda = -1,\ \lambda = 3}$ are the eigenvalues of $M = \begin{pmatrix} 1 & 4 \\ 1 & 1 \end{pmatrix}$

* Find eigenvectors of $\lambda = -1$:

Solve $\left(M - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right)\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ for $\begin{pmatrix} v_1 \\ v_2 \end{pmatrix}$

$$\left(M - (-1)\begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right)\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 1+1 & 4 \\ 1 & 1+1 \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

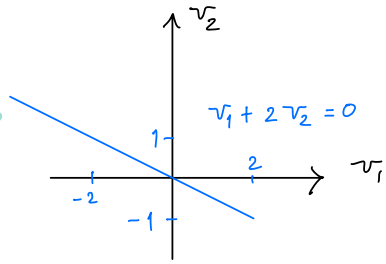
$$\left.\begin{aligned} 2v_1 + 4v_2 &= 0 \\ 1v_1 + 2v_2 &= 0 \end{aligned}\right\}$$

Fact from linear algebra :

Because $\det\left(M - \lambda \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}\right) = 0$, these two equations are equivalent.

If they are not, double check your computation

$v_1 + 2v_2 = 0$



Any point / vector on this line is an <u>eigenvector</u> of $\lambda = -1$.

To find one eigenvector, just pick any point on this line (except the origin),

e.g. $v = \begin{pmatrix} -2 \\ 1 \end{pmatrix}$ or $\begin{pmatrix} 1 \\ -\frac{1}{2} \end{pmatrix}$

Remark: Eigenvectors are not unique. In fact, if v is an eigenvector of λ, then so is cv for any nonzero number c.

* Find eigenvectors of $\lambda = 3$ for $M = \begin{pmatrix} 1 & 4 \\ 1 & 1 \end{pmatrix}$:

Solve $(M - \lambda I)v = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$ $\qquad I := \begin{pmatrix} 1 & 0 \\ 0 & 1 \end{pmatrix}$

$$\begin{pmatrix} 1-3 & 4-0 \\ 1-0 & 1-3 \end{pmatrix} v = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} -2 & 4 \\ 1 & -2 \end{pmatrix} \begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{aligned} -2v_1 + 4v_2 &= 0 \\ v_1 - 2v_2 &= 0 \end{aligned}$$

Both equations describe the same line

Take the point $v_2 = 1$, $v_1 = 2v_2 = 2$: $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$

(or any point in this line except the origin).

Then $\begin{pmatrix} 2 \\ 1 \end{pmatrix}$ is an eigenvector of M associated to $\lambda = 3$