The eigenvalue method for homogeneous system part b

THEOREM 1 Eigenvalue Solutions of $\mathbf{x}' = \mathbf{A}\mathbf{x}$

Let λ be an eigenvalue of the [constant] coefficient matrix $\mathbf{A}$ of the first-order linear system

size $n \times n$

$$\frac{d\mathbf{x}}{dt} = \mathbf{A}\mathbf{x}.$$

If $\mathbf{v}$ is an eigenvector associated with λ, then

$$\mathbf{x}(t) = \mathbf{v}e^{\lambda t}$$

entries not all zeros

is a nontrivial solution of the system.

The Eigenvalue Method

Step 1: Find all eigenvalues of A.

We will consider only the case where the eigenvalues are all distinct.

Step 2: For each eigenvalue λ, find an eigenvector $\vec{v}$.

There are always infinitely many eigenvectors, but it's enough to pick one

This gives us n "linearly independent" vectors.

Step 3: Each eigenvalue λ with eigenvector v contributes one term to the general solution of $x' = Ax$:

* Each real eigenvalue λ with eigenvector v contributes a term $c_1 e^{\lambda t} v$

* Each complex conjugate pair $\lambda = p + qi$ and $\bar{\lambda} = p - qi$ with eigen vectors $v = \begin{bmatrix} a_1 + b_1 i \\ a_2 + b_2 i \\ \vdots \\ a_n + b_n i \end{bmatrix}$ and $\bar{v} = \begin{bmatrix} a_1 - b_1 i \\ a_2 - b_2 i \\ \vdots \\ a_n - b_n i \end{bmatrix}$, respectively,

contributes two terms $c_1 e^{\lambda t} v + c_2 e^{\bar{\lambda} t} \bar{v}$

Remark: These are complex-valued functions.

We want the real-valued terms only

(For simplicity, I will demonstate this assuming A is 2×2.)

Let $v=\begin{bmatrix} a+bi \\ c+di \end{bmatrix}$, so $\bar{v}=\begin{bmatrix} a-bi \\ c-di \end{bmatrix}$

Recall:

$$e^{(p+qi)t} = e^{pt}e^{iqt}$$

$$= e^{pt}\left(\cos(qt)+i\sin(qt)\right)$$

Rewrite

$$e^{\lambda t}v = e^{(p+qi)t}\left(\begin{bmatrix} a \\ c \end{bmatrix} + i\begin{bmatrix} b \\ d \end{bmatrix}\right)$$

$$= e^{pt}\left(\cos(qt)+i\sin(qt)\right)\left(\begin{bmatrix} a \\ c \end{bmatrix} + i\begin{bmatrix} b \\ d \end{bmatrix}\right)$$

$$\overset{(\text{foil})}{=} \underbrace{e^{pt}\left(\cos(qt)\begin{bmatrix} a \\ c \end{bmatrix} - \sin(qt)\begin{bmatrix} b \\ d \end{bmatrix}\right)}_{\text{real part}} + \underbrace{e^{pt}\, i\left(\cos(qt)\begin{bmatrix} b \\ d \end{bmatrix} + \sin(qt)\begin{bmatrix} a \\ c \end{bmatrix}\right)}_{\text{imaginary part}}$$

We get two linearly independent real-valued solutions:

$$e^{pt}\left(\cos(qt)\begin{bmatrix} a \\ c \end{bmatrix} - \sin(qt)\begin{bmatrix} b \\ d \end{bmatrix}\right) \text{ and } e^{pt}\left(\cos(qt)\begin{bmatrix} b \\ d \end{bmatrix} + \sin(qt)\begin{bmatrix} a \\ c \end{bmatrix}\right)$$

Rem: You can use either eigenvalue $p+qi$ or $p-qi$.

Ex: Find a general solution of the system

$$\frac{dx_1}{dt} = 4x_1 - 3x_2,$$

$$\frac{dx_2}{dt} = 3x_1 + 4x_2.$$

Ans: The matrix form of this system is $\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \underbrace{\begin{bmatrix} 4 & -3 \\ 3 & 4 \end{bmatrix}}_{A:=} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$

(Step 1) Previously we computed that the eigenvalues of A are $4-3i$ and $4+3i$

(Step 2) Previously we found an eigenvector of $4-3i$ which is $\begin{bmatrix} 1 \\ i \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix} + i \begin{bmatrix} 0 \\ 1 \end{bmatrix}$

Extra info that we don't need

(and an eigenvector of $4+3i$ which is $\begin{bmatrix} 1 \\ 0 \end{bmatrix} + i \begin{bmatrix} 0 \\ -1 \end{bmatrix}$)

(Step 3) We get two real-valued linearly independent solutions

$$e^{4t}\left(\cos(-3t) \begin{bmatrix} 1 \\ 0 \end{bmatrix} - \sin(-3t) \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) \quad \text{and} \quad e^{4t}\left(\cos(-3t) \begin{bmatrix} 0 \\ 1 \end{bmatrix} + \sin(-3t) \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right)$$

Recall: $\cos(-x) = \cos(x)$ and $\sin(-x) = -\sin(x)$
$\cos(-3t) = \cos(3t)$, $-\sin(-3t) = \sin(3t)$

A real-valued general solution is

$$\vec{x}(t) = C_1 e^{4t}\left(\cos(3t) \begin{bmatrix} 1 \\ 0 \end{bmatrix} + \sin(3t) \begin{bmatrix} 0 \\ 1 \end{bmatrix} \right) + C_2 e^{4t}\left(\cos(3t) \begin{bmatrix} 0 \\ 1 \end{bmatrix} - \sin(3t) \begin{bmatrix} 1 \\ 0 \end{bmatrix} \right)$$

$$= C_1 e^{4t} \begin{bmatrix} \cos(3t) \\ \sin(3t) \end{bmatrix} + C_2 e^{4t} \begin{bmatrix} -\sin(3t) \\ \cos(3t) \end{bmatrix}$$

$$= \begin{bmatrix} C_1 e^{4t} \cos(3t) - C_2 e^{4t} \sin(3t) \\ C_1 e^{4t} \sin(3t) + C_2 e^{4t} \cos(3t) \end{bmatrix} \quad \text{for } C_1, C_2 \in \mathbb{R}$$

(in matrix form)

You can also write

$$x_1(t) = e^{4t}(c_1 \cos 3t - c_2 \sin 3t),$$
$$x_2(t) = e^{4t}(c_1 \sin 3t + c_2 \cos 3t), \quad \text{for } C_1, C_2 \in \mathbb{R}.$$

Ex: Find the (unique) solution to the IVP

$$\frac{dx_1}{dt} = 4x_1 - 3x_2,$$

$$\frac{dx_2}{dt} = 3x_1 + 4x_2.$$

$$\underbrace{\begin{bmatrix} x_1(0) \\ x_2(0) \end{bmatrix} = \begin{bmatrix} 1 \\ 0 \end{bmatrix}}_{\text{initial condition}}$$

Ans We computed a general solution to the system $\begin{bmatrix} x_1' \\ x_2' \end{bmatrix} = \begin{bmatrix} 4 & -3 \\ 3 & 4 \end{bmatrix} \begin{bmatrix} x_1 \\ x_2 \end{bmatrix}$:

$$x_1(t) = e^{4t}(c_1 \cos 3t - c_2 \sin 3t),$$
$$x_2(t) = e^{4t}(c_1 \sin 3t + c_2 \cos 3t), \qquad \text{for } C_1, C_2 \in \mathbb{R}.$$

Impose the initial conditions:

$$\left.\begin{aligned} 1 &= x_1(0) = e^0 \left(c_1 \cos(0) - C_2 \sin(0)\right) \\ 0 &= x_2(0) = e^0 \left(C_1 \sin(0) + C_2 \cos(0)\right) \end{aligned}\right\} \Rightarrow \begin{aligned} 1 &= C_1 \\ 0 &= C_2 \end{aligned}$$

The solution to the IVP is $\boxed{\begin{aligned} x_1(t) &= e^{4t} \cos(3t) \\ x_2(t) &= e^{4t} \sin(3t) \end{aligned}}$.