Sec 5.2 part a: Eigenvalues and eigenvectors, revisited

- **Definition** Let M be an $n \times n$ matrix with real entries

  A number λ (lambda) is an eigenvalue of M if there exists

  a nonzero vector v ($n \times 1$ matrix) such that $Av = \lambda v$.

  Such nonzero vector v is called an eigenvector of M.

- **Method for finding all eigenvalues of a matrix M:**

  Write down the characteristic equation of M

  $$\det(M - \lambda I) = 0.$$

  The roots (real or non-real complex) are the eigenvalues of M.

- Method for finding all eigenvalues of a 2×2 matrix $M = \begin{bmatrix} a & b \\ c & d \end{bmatrix}$

  $a, b, c, d \in \mathbb{R}$

  The characteristic equation of M is

  $$\det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} - \begin{bmatrix} \lambda & 0 \\ 0 & \lambda \end{bmatrix}\right) = 0$$

  $$\begin{vmatrix} a-\lambda & b \\ c & d-\lambda \end{vmatrix} = 0$$

  simplified

  $$(a-\lambda)(d-\lambda) - bc = 0$$

  The roots of this characteristic equation are the eigenvalues of M.

There are 3 possible cases:

1. The characteristic equation has two distinct real eigenvalues
2. The characteristic equation has a pair of complex conjugate eigenvalues $\lambda = p + qi$ and $\bar{\lambda} = p - qi$
3. The characteristic equation has one real root with multiplicity 2.

Example: Find all eigenvalues of the matrix $M = \begin{bmatrix} 4 & -3 \\ 3 & 4 \end{bmatrix}$

Answer: The characteristic equation of M is

$$\det\left(M - \lambda \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right) = 0$$

$$\begin{vmatrix} 4-\lambda & -3 \\ 3 & 4-\lambda \end{vmatrix} = 0$$

$$(4-\lambda)^2 - 3(-3) = 0$$

$$(4-\lambda)^2 + 9 = 0$$

$$(4-\lambda)^2 = -9$$

$$4-\lambda = \pm 3i$$

$\lambda = 4 \pm 3i$ are the roots of the characteristic equation of M

So M has two eigenvalues $\boxed{\lambda_1 = 4-3i \text{ and } \lambda_2 = 4+3i.}$

Example: Find at least one eigenvector for each of the eigenvalues of the matrix $M = \begin{bmatrix} 4 & -3 \\ 3 & 4 \end{bmatrix}$.

Answer: To find one eigenvector for $\lambda_1 = 4-3i$,

we need to find $\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ which satisfies

$$\left(M - \lambda_1 \begin{bmatrix} 1 & 0 \\ 0 & 1 \end{bmatrix}\right)\begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}:$$

$$\begin{pmatrix} 4-(4-3i) & -3 \\ 3 & 4-(4-3i) \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

$$\begin{pmatrix} 3i & -3 \\ 3 & 3i \end{pmatrix}\begin{pmatrix} v_1 \\ v_2 \end{pmatrix} = \begin{pmatrix} 0 \\ 0 \end{pmatrix}$$

You can
to the last step

$$\begin{bmatrix} 3i\,v_1 - 3\,v_2 \\ 3\,v_1 + 3i\,v_2 \end{bmatrix} = \begin{bmatrix} 0 \\ 0 \end{bmatrix}$$

$$3i\,v_1 - 3\,v_2 = 0$$

$$3\,v_1 + 3i\,v_2 = 0$$

Optional sanity check: the two equations should be equivalent — if not, your λ_1 is not an eigenvalue or your arithmetic is wrong.

Yes, the two equations are equivalent.

Multiply the top equation by $-i$ to get the bottom equation:

$$(-i)(3i\, v_1 - 3v_2) = (-i)\, 0$$

$$-i^2\, 3\, v_1 + 3i\, v_2 = 0$$

$$3\, v_1 + 3i\, v_2 = 0$$

The equations tell us that $v_2 = i\, v_1$.

Pick any nonzero number for v_1, e.g. set $v_1 = 1$, then $v_2 = i$

So an eigenvector for $\lambda_1 = 4-3i$ is $\boxed{\begin{pmatrix} 1 \\ i \end{pmatrix}}$.

Other eigenvectors for $\lambda_1 = 4-3i$ are $\begin{pmatrix} i \\ -1 \end{pmatrix}$, $\begin{pmatrix} 2 \\ 2i \end{pmatrix}$, $\begin{pmatrix} 6+3i \\ 6i-3 \end{pmatrix}$.

Fact: If a 2x2 matrix has a pair of complex conjugate eigenvalues

$p+qi$ and $p-qi$,

you only need to find eigenvectors for one of the eigenvalues — the eigenvectors for the other eigenvalue are obtained by conjugation.

If $v = \begin{bmatrix} a+bi \\ c+di \end{bmatrix}$ is an eigenvector for eigenvalue $\lambda = p+qi$,

then $\bar{v} = \begin{bmatrix} a-bi \\ c-di \end{bmatrix}$ is an eigenvector for $\bar{\lambda} = p-qi$.

So an eigenvector for $\lambda_2 = 4+3i$ is $\boxed{\begin{pmatrix} 1 \\ -i \end{pmatrix}}$. ← In class we computed this directly

The eigenvalue method for homogeneous system part a

<u>Thm</u> (Eigenvalue solutions of $x' = Ax$)

Consider the 1st-order linear system of ODEs

$$\vec{x}'(t) = \underbrace{A}_{\text{matrix w/ constant coefficients}} \vec{x}(t) \qquad \left(\text{or} \quad \frac{d}{dt}\vec{x}(t) = A\,\vec{x}(t)\right).$$

If λ is an eigenvalue of A and

$\vec{v}$ is an eigenvector of A associated with λ,

then $\vec{x}(t) = \vec{v}\, e^{\lambda t}$ is a solution of the system.

Explanation: Consider our previous running example,

a homogeneous linear system $\vec{x}'(t) = \begin{bmatrix} 1 & 4 \\ 1 & 1 \end{bmatrix} \vec{x}(t)$

we expect a solution of the form $\vec{x}(t) = e^{rt} v$

Guess a solution $\vec{x}(t) = e^{rt} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$

$$\vec{x}'(t) = r e^{rt} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

Substitute $\vec{x}(t)$ and $\vec{x}'(t)$ into the system of ODEs (equivalently, matrix equation)

$$r e^{rt} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ 1 & 1 \end{bmatrix} e^{rt} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

$$r \begin{bmatrix} v_1 \\ v_2 \end{bmatrix} = \begin{bmatrix} 1 & 4 \\ 1 & 1 \end{bmatrix} \begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$$

To find r and $\begin{bmatrix} v_1 \\ v_2 \end{bmatrix}$ is the same as finding

an eigenvector v of $\begin{bmatrix} 1 & 4 \\ 1 & 1 \end{bmatrix}$ and an eigenvalue corresponding to v.

Example: Find a general solution of the system

$$x_1' = x_1 + 4x_2$$

$$x_2' = x_1 + x_2$$

Answer: The matrix form of the system is

$$\vec{x}' = \begin{bmatrix} 1 & 4 \\ 1 & 1 \end{bmatrix} \vec{x}$$

Earlier, we computed that the matrix $\begin{bmatrix} 1 & 4 \\ 1 & 1 \end{bmatrix}$

has two distinct eigenvalues

-1 with an eigenvector $\begin{bmatrix} -2 \\ 1 \end{bmatrix}$ among infinitely many choices

3 with an eigenvector $\begin{bmatrix} 2 \\ 1 \end{bmatrix}$. among many

* So $\vec{x}(t) = e^{-t}\begin{bmatrix} -2 \\ 1 \end{bmatrix}$ is a solution of $\vec{x}'(t) = \begin{bmatrix} 1 & 4 \\ 1 & 1 \end{bmatrix} \vec{x}(t)$,

and $\vec{x}(t) = e^{3t}\begin{bmatrix} 2 \\ 1 \end{bmatrix}$ is also a solution of $\vec{x}'(t) = \begin{bmatrix} 1 & 4 \\ 1 & 1 \end{bmatrix} \vec{x}(t)$.

* The general solution of $\vec{x}'(t) = \begin{bmatrix} 1 & 4 \\ 1 & 1 \end{bmatrix} \vec{x}(t)$ is

a linear combination of

two linearly independent solutions,

$$\boxed{\vec{x}(t) = c_1 e^{-t}\begin{bmatrix} -2 \\ 1 \end{bmatrix} + c_2 e^{3t}\begin{bmatrix} 2 \\ 1 \end{bmatrix} \quad \text{for } c_1, c_2 \in \mathbb{R}}$$