5.1 Matrices and linear systems

- A matrix-valued function (or simply matrix function) is a matrix whose entries are functions of t.

Ex: $\vec{x}(t) = \begin{bmatrix} x_1(t) \\ x_2(t) \\ x_3(t) \end{bmatrix}$, $\vec{A}(t) = \begin{bmatrix} a_{11}(t) & a_{12}(t) & a_{13}(t) \\ a_{21}(t) & a_{22}(t) & a_{23}(t) \end{bmatrix} = (a_{ij}(t))_{\substack{1 \le i \le 2, \\ 1 \le j \le 3}}$

- A matrix function is continuous at a point (or an interval) if each of its entries is.

- A matrix function is differentiable at a point (or an interval) if each of its entries is.

Ex: $\vec{x}(t) = \begin{pmatrix} e^t \\ e^{2t} \\ 7 \end{pmatrix}$ $\qquad$ $\vec{A}(t) = \begin{bmatrix} 0 & \sin(t) \\ \cos(t) & \tan(2t) \end{bmatrix}$

$\vec{x}(t)$ is continuous and differentiable on $\mathbb{R}$ $\qquad$ $\vec{A}(t)$ is continuous and differentiable on $(-\frac{\pi}{4}, \frac{\pi}{4})$

- The derivative of a differentiable matrix function $A(t) = (a_{ij}(t))$ is $A'(t) = \left(\frac{d}{dt} a_{ij}(t) \right)$ (differentiate each entry)

Ex: $\vec{x}\,'(t) = \begin{pmatrix} e^t \\ 2e^{2t} \\ 0 \end{pmatrix}$ $\qquad$ $A'(t) = \begin{pmatrix} 0 & \cos(t) \\ -\sin(t) & \frac{2}{\cos^2(t)} \end{pmatrix}$

1st-order linear systems

- A system of 1st-order ODEs is <u>linear</u> if it can be written as

column vectors

$$\vec{x}'(t) = \underbrace{P(t)}_{\text{rectangular matrix function}} \vec{x}(t) + \underbrace{f(t)}_{\text{matrix (column vector) function}}$$

Ex :

$$\begin{bmatrix} x'(t) \\ y'(t) \end{bmatrix} \overset{(*)}{=} \underbrace{\begin{bmatrix} \sin(t) & t^2 \\ 0 & \cos(t) \end{bmatrix}}_{P(t)} \begin{bmatrix} x(t) \\ y(t) \end{bmatrix} + \underbrace{\begin{bmatrix} \cos(t) \\ e^t \end{bmatrix}}_{f(t)}$$

Here $P(t) = \begin{bmatrix} \sin(t) & t^2 \\ 0 & \cos(t) \end{bmatrix}$ and $f(t) = \begin{bmatrix} \cos(t) \\ e^t \end{bmatrix}$

The equation (*) is equivalent to the system of two equations

$$\begin{cases} x'(t) = \sin(t)\ x(t) + t^2\ y(t) + \cos(t) \\ y'(t) = 0\ x(t) + \cos(t)\ y(t) + e^t \end{cases}$$

Ex : The 1st-order system

$$\begin{cases} x_1' = 4x_1 - 3x_2 \\ x_2' = 6x_1 - 7x_2 \end{cases}$$

can be written in matrix form as

$$\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} 4 & -3 \\ 6 & -7 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$$

Def:

A solution of the 1st-order linear system

$$\vec{x}\,'(t) = \overbrace{P(t)}^{n\times n}\,\vec{x}(t) + \vec{f}(t)$$

vectors of height n

is a family of n functions $x_1(t), x_2(t), \dots, x_n(t)$ that satisfy the equality

$$\begin{pmatrix} x_1'(t) \\ x_2'(t) \\ \vdots \\ x_n'(t) \end{pmatrix} = P(t) \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix} + \vec{f}(t)$$

Ex of Def:

Is $\vec{x}_a(t) = \begin{bmatrix} 3e^{2t} \\ 2e^{2t} \end{bmatrix}$ a solution to the system $\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} 4 & -3 \\ 6 & -7 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$

Is $\vec{x}_b(t) = \begin{pmatrix} e^{-5t} \\ 3e^{-5t} \end{pmatrix}$ a solution ?

Answer: Compute $\vec{x}_a'(t) = \begin{pmatrix} 6e^{2t} \\ 4e^{2t} \end{pmatrix}$.

Substitute both $\vec{x}_a(t)$ and $\vec{x}_a'(t)$ into the system:

$$\text{LHS} = \begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} 6e^{2t} \\ 4e^{2t} \end{pmatrix}$$

$$\text{RHS} = \begin{bmatrix} 4 & -3 \\ 6 & -7 \end{bmatrix}\begin{bmatrix} 3e^{2t} \\ 2e^{2t} \end{bmatrix} = \begin{bmatrix} (12-6)e^{2t} \\ (18-14)e^{2t} \end{bmatrix} = \begin{bmatrix} 6e^{2t} \\ 4e^{2t} \end{bmatrix}$$

Since LHS = RHS, $\vec{x}_a(t)$ is a solution of the system

Homework: Verify that $\vec{x}_b(t)$ is also a solution of the system.

Remark: Every n-th order linear equation is equivalent to a 1st-order linear system of n equations in n variables.

Ex of Remark:

Convert the 2nd-order linear equation

$$y'' + 2y' + 3y = \sin(t)$$

into a 1st-order linear system.

Tip:
In Webwork, you have

$u'' + 2u' + 3u = \sin(t)$

Set $u = u$
$v = u'$

Ans: Let $u = y(t)$
$v = y'(t)$

Then $u' = y'(t) = v$

$$v' = y'' = \sin(t) - 2y' - 3y$$
$$= \sin(t) - 2v - 3u$$

$$\begin{cases} u' = v \\ v' = -3u - 2v + \sin(t) \end{cases}$$

or

$$\begin{pmatrix} u' \\ v' \end{pmatrix} = \begin{pmatrix} 0 & 1 \\ -3 & -2 \end{pmatrix} \begin{pmatrix} u \\ v \end{pmatrix} + \begin{bmatrix} 0 \\ \sin(t) \end{bmatrix}$$

is the 1st-order system equivalent to the original 1st-order ODE.

Thm (Existence and uniqueness of solutions of 1st-order linear systems)

Consider

$$\begin{pmatrix} x_1'(t) \\ x_2'(t) \\ \vdots \\ x_n'(t) \end{pmatrix} = P(t) \begin{pmatrix} x_1(t) \\ x_2(t) \\ \vdots \\ x_n(t) \end{pmatrix} + \begin{pmatrix} f_1(t) \\ f_2(t) \\ \vdots \\ f_n(t) \end{pmatrix}$$

Suppose each function in $P(t)$ and $f_1(t), f_2(t), \dots, f_n(t)$ are continuous in an open interval I containing the point a

Then, given numbers $b_1, b_2, \dots, b_n \in \mathbb{R}$, the system has a unique solution on I that satisfies the initial conditions

$$x_1(a) = b_1,\ x_2(a) = b_2,\ \dots,\ x_n(a) = b_n.$$

Def: Consider the 1st-order linear system

$$\vec{x}'(t) = P(t)\,\vec{x}(t) + \vec{f}(t)$$

The associated homogeneous system is

$$\vec{x}'(t) = P(t)\,\vec{x}(t)$$

Thm (Principle of superposition for homogeneous system)

IF $\vec{x}_a(t)$ and $\vec{x}_b(t)$ are both solutions in I of the homogeneous linear system $\vec{x}'(t) = P(t)\,\vec{x}(t)$,

THEN

any linear combination

$$\vec{x}(t) = c_1\,\vec{x}_a(t) + c_2\,\vec{x}_b(t), \quad \text{for } c_1, c_2 \in \mathbb{R}$$

is also a solution of $\vec{x}'(t) = P(t)\,\vec{x}(t)$ on I.

Ex of Thm:

From the previous ex, we see that

$\vec{x}_a(t) = \begin{pmatrix} 3e^{2t} \\ 2e^{2t} \end{pmatrix}$ and $\vec{x}_b(t) = \begin{pmatrix} e^{-5t} \\ 3e^{-5t} \end{pmatrix}$ are both solutions

of the homogeneous system $\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} 4 & -3 \\ 6 & -7 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ in $\mathbb{R}$.

So the thm (principle of superposition) tells us that

$$c_1\,\vec{x}_a(t) + c_2\,\vec{x}_b(t) = \begin{pmatrix} 3c_1 e^{2t} + c_2 e^{-5t} \\ 2c_1 e^{2t} + 3c_2 e^{-5t} \end{pmatrix}, \quad \text{for any } c_1, c_2$$

is also a solution in $\mathbb{R}$

Note: In this example, $I = \mathbb{R}$

Def: Two vector-valued functions $\vec{x}_a(t)$ and $\vec{x}_b(t)$ are <u>linearly dependent</u> on the interval I

if there exists a nonzero constant c such that

$$c\,\vec{x}_a(t) = \vec{x}_b(t) \quad \text{OR} \quad \vec{x}_a(t) = c\,\vec{x}_b(t).$$

Otherwise, $\vec{x}_a(t)$ and $\vec{x}_b(t)$ are <u>linearly dependent.</u>

Ex: $\vec{x}_a = \begin{bmatrix} 3e^{t} \\ e^{t} \end{bmatrix}$ and $\vec{x}_b = \begin{bmatrix} -6e^{t} \\ -2e^{t} \end{bmatrix}$ are linearly dependent

because $-2\vec{x}_a = \vec{x}_b$

Ex: $\vec{x}_a(t) = \begin{pmatrix} 3e^{2t} \\ 2e^{2t} \end{pmatrix}$ and $\vec{x}_b(t) = \begin{pmatrix} e^{-5t} \\ 3e^{-5t} \end{pmatrix}$ are linearly independent.

Def: The <u>Wronskian</u> of $\vec{x}_a(t) = \begin{bmatrix} x_{1a}(t) \\ x_{2a}(t) \end{bmatrix}$ and $\vec{x}_b(t) = \begin{bmatrix} x_{1b}(t) \\ x_{2b}(t) \end{bmatrix}$

is
$$W(\vec{x}_a(t), \vec{x}_b(t)) = \det\left(\begin{bmatrix} x_{1a}(t) & x_{1b}(t) \\ x_{2a}(t) & x_{2b}(t) \end{bmatrix} \right)$$

this column is $\vec{x}_a(t)$ — this column is $\vec{x}_b(t)$

For short, write

$$\begin{vmatrix} a & b \\ c & d \end{vmatrix} := \det\left(\begin{bmatrix} a & b \\ c & d \end{bmatrix} \right)$$

$$= x_{1a}(t)\, x_{2b}(t) - x_{2a}(t)\, x_{1b}(t)$$

Thm (Wronskian & linear independence):
If $W(\vec{u}(t), \vec{v}(t))$ is not the zero function,
then $\vec{u}(t)$ and $\vec{v}(t)$ are linearly independent.

Ex of Thm:

$\vec{x}_a(t) = \begin{pmatrix} 3e^{2t} \\ 2e^{2t} \end{pmatrix}$ and $\vec{x}_b(t) = \begin{pmatrix} e^{-5t} \\ 3e^{-5t} \end{pmatrix}$

$$W(\vec{x}_a, \vec{x}_b) = \begin{vmatrix} 3e^{2t} & e^{-5t} \\ 2e^{2t} & 3e^{-5t} \end{vmatrix} = 9e^{2t}e^{-5t} - 2e^{2t}e^{-5t}$$
$$= 9e^{-3t} - 2e^{-3t}$$
$$= 7e^{-3t}$$

This is not the zero function (e.g. the value at $t=0$ is $7 \neq 0$),
so the thm (Wronskian & linear independence)
tells us that $\vec{x}_a(t)$ and $\vec{x}_b(t)$ are linearly independent.

Thm (General solutions of homogeneous systems)

Suppose $\vec{x}_1(t), \vec{x}_2(t), \ldots, \vec{x}_n(t)$ are linearly independent solutions (on an open interval I)

of the homogeneous first-order linear system

with n equations in n variables

$$\vec{x}'(t) = P(t)\,\vec{x}(t) \qquad (*)$$

Suppose $P(t)$ is continuous on I.

Then any solution $\vec{x}(t)$ of the system $(*)$ can be written

as $\vec{x}(t) = c_1 \vec{x}_1(t) + c_2 \vec{x}_2(t) + \ldots + c_n \vec{x}_n(t)$ for some $c_1, c_2, \ldots, c_n \in \mathbb{R}$

for all t in I.

(This means, to find a general solution, we only have to find n linearly independent solutions.)

Ex of Thm:

From the previous ex, we see that

$\vec{x}_a(t) = \begin{pmatrix} 3e^{2t} \\ 2e^{2t} \end{pmatrix}$ and $\vec{x}_b(t) = \begin{pmatrix} e^{-5t} \\ 3e^{-5t} \end{pmatrix}$ are linearly independent solutions

of the homogeneous system $\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} 4 & -3 \\ 6 & -7 \end{pmatrix} \begin{pmatrix} x_1 \\ x_2 \end{pmatrix}$ in $\mathbb{R}$.

By Thm (General solutions of homogeneous systems),

since $P(t) = \begin{pmatrix} 4x_1 - 3x_2 \\ 6x_1 - 7x_2 \end{pmatrix}$ 2 equations in 2 variables

a general solution of the system is

$$\vec{x}(t) = c_1 \begin{pmatrix} 3e^{2t} \\ 2e^{2t} \end{pmatrix} + c_2 \begin{pmatrix} e^{-5t} \\ 3e^{-5t} \end{pmatrix} = \begin{pmatrix} c_1 3e^{2t} + c_2 e^{-5t} \\ c_1 2e^{2t} + c_2 3e^{-5t} \end{pmatrix} \quad \text{for } c_1, c_2 \in \mathbb{R}.$$

Thm (General solution of $\underline{\text{non}}$ homogeneous systems)

Let $\vec{x}'(t) = P(t)\,\vec{x}(t) + \underset{\substack{\Uparrow \\ \text{not all } 0\text{s}}}{\vec{f}(t)}$ be a $\underline{\text{non}}$ homogeneous 1st-order linear system of n equations in n variables.

Then its general solution is

$$\left(\begin{array}{c}\text{General solution} \\ \text{of the} \\ \text{homogeneous} \\ \text{system} \\ \vec{x}'(t) = P(t)\,\vec{x}(t)\end{array}\right) + \left(\begin{array}{c}\text{Particular solution} \\ \text{of the (original)} \\ \text{non homogeneous} \\ \text{system}\end{array}\right)$$

Ex of Thm (General solution of $\underline{\text{non}}$ homogeneous systems)

Example 9 The nonhomogeneous linear system

$$\begin{aligned} x_1' &= 3x_1 - 2x_2 \phantom{{}-2x_3} - 9t + 13, \\ x_2' &= -x_1 + 3x_2 - 2x_3 + 7t - 15, \\ x_3' &= - x_2 + 3x_3 - 6t + 7 \end{aligned}$$

is of the form in $\dfrac{d\mathbf{x}}{dt} = \mathbf{P}(t)\mathbf{x} + \mathbf{f}(t)$, with

$$\mathbf{P}(t) = \begin{bmatrix} 3 & -2 & 0 \\ -1 & 3 & -2 \\ 0 & -1 & 3 \end{bmatrix}, \quad \mathbf{f}(t) = \begin{bmatrix} -9t + 13 \\ 7t - 15 \\ -6t + 7 \end{bmatrix}.$$

A general solution of the associated homogeneous linear system

$$\frac{d\mathbf{x}}{dt} = \begin{bmatrix} 3 & -2 & 0 \\ -1 & 3 & -2 \\ 0 & -1 & 3 \end{bmatrix}\mathbf{x}$$

is given by

$$\mathbf{x}_c(t) = \begin{bmatrix} 2c_1e^t + 2c_2e^{3t} + 2c_3e^{5t} \\ 2c_1e^t \phantom{{}+ 2c_2e^{3t}} - 2c_3e^{5t} \\ c_1e^t - c_2e^{3t} + c_3e^{5t} \end{bmatrix},$$

and we can verify by substitution that the function

$$\mathbf{x}_p(t) = \begin{bmatrix} 3t \\ 5 \\ 2t \end{bmatrix}$$

(found using a computer algebra system, or perhaps by a human being using a method discussed in Section 5.7) is a particular solution of the original nonhomogeneous system. Consequently, Theorem 4 implies that a general solution of the nonhomogeneous system is given by (textbook)

$$\mathbf{x}(t) = \mathbf{x}_c(t) + \mathbf{x}_p(t);$$

that is, by

$$\begin{aligned} x_1(t) &= 2c_1e^t + 2c_2e^{3t} + 2c_3e^{5t} + 3t, \\ x_2(t) &= 2c_1e^t \phantom{{}+ 2c_2e^{3t}} - 2c_3e^{5t} + 5, \\ x_3(t) &= c_1e^t - c_2e^{3t} + c_3e^{5t} + 2t. \end{aligned}$$

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Initial value problems

Ex:

Solve the initial value problem (IVP)

$$\begin{pmatrix} x_1' \\ x_2' \end{pmatrix} = \begin{pmatrix} 4 & -3 \\ 6 & -7 \end{pmatrix}\begin{pmatrix} x_1 \\ x_2 \end{pmatrix}, \quad \vec{x}(0) = \begin{bmatrix} \vec{x}_1(0) \\ \vec{x}_2(0) \end{bmatrix} = \begin{bmatrix} 0 \\ 2 \end{bmatrix}$$

Answer:

Previously, we said that

a general solution of the system is

$$\begin{bmatrix} \vec{x}_1(t) \\ \vec{x}_2(t) \end{bmatrix} = c_1 \begin{pmatrix} 3e^{2t} \\ 2e^{2t} \end{pmatrix} + c_2 \begin{pmatrix} e^{-5t} \\ 3e^{-5t} \end{pmatrix} = \begin{pmatrix} c_1 3e^{2t} + c_2 e^{-5t} \\ c_1 2e^{2t} + c_2 3e^{-5t} \end{pmatrix} \quad \text{for } c_1, c_2 \in \mathbb{R}.$$

Impose the initial condition (to the general solution):

$$\begin{bmatrix} 0 \\ 2 \end{bmatrix} = \begin{bmatrix} \vec{x}_1(0) \\ \vec{x}_2(0) \end{bmatrix} = \begin{pmatrix} c_1 3e^{0} + c_2 e^{0} \\ c_1 2e^{0} + c_2 3e^{0} \end{pmatrix}$$

So $\left.\begin{aligned} c_1 3 + c_2 &= 0 \\ c_1 2 + c_2 3 &= 2 \end{aligned}\right\} \Rightarrow$

$$c_2 = -3c_1$$

$$c_1 2 + (-3c_1) 3 = 2$$

$$c_1(2-9) = 2$$

$$\boxed{c_1 = -\frac{2}{7}} \qquad c_2 = \boxed{\frac{6}{7}}$$

The solution to the IVP is

$$\left.\begin{aligned} \vec{x}_1(t) &= -\frac{2}{7} 3e^{2t} + \frac{6}{7} e^{-5t} \\ \vec{x}_2(t) &= -\frac{2}{7} 2e^{2t} + \frac{6}{7} 3e^{-5t} \end{aligned}\right\} \text{ or } \quad \vec{x}(t) = -\frac{2}{7}\begin{pmatrix} 3e^{2t} \\ 2e^{2t} \end{pmatrix} + \frac{6}{7}\begin{pmatrix} e^{-5t} \\ 3e^{-5t} \end{pmatrix}$$