Sec 3.5 Nonhomogeneous linear equation with constant coefficients

Consider a nonhomogeneous linear ODE with constant coefficients

$$a_n y^{(n)} + \dots + a_2 y'' + a_1 y' + a_0 y = f(x) \qquad [N]$$

where $a_n, \dots, a_2, a_1, a_0 \in \mathbb{R}$ are constants and $a_n \neq 0$.

Then

$$a_n y^{(n)} + \dots + a_2 y'' + a_1 y' + a_0 y = 0 \qquad [H]$$

is the corresponding homogeneous ODE.

Fact 1 (from Sec 3.1)

(General solution $y(x)$ of the nonhomogeneous ODE [N]) = (General solution $y_c(x)$ of the homogeneous ODE [H]) + (a particular solution $y_p(x)$ of the nonhomogeneous ODE [N])

Use the characteristic equation method from Sec 3.3

More complicated. We'll learn a method that works for a few cases

Note: $y_c(x)$ is called the complementary solution of [N]

or the homogeneous solution of [N]

Ex 1 Find a general solution of $y'' - 5y' + 6y = \underbrace{2xe^{3x}}_{f(x)}$ [N]

Step 1 Find a general solution of the homogeneous ODE

$$y'' - 5y' + 6y = 0 \quad [H]$$

The characteristic equation is $r^2 - 5r + 6 = 0$ [C]

$(r-2)(r-3) = 0$

2, 3 are roots, each w/ multiplicity 1

(From Sec 3.3) A general solution of [H] is

$$y_c(x) = C_1 e^{2x} + C_2 e^{3x}, \quad C_1, C_2 \in \mathbb{R}$$

Step 2 The nonhomogeneous part of [N] is

$$f(x) = \underbrace{e^{3x}}_{} \underbrace{2x}_{}$$

Like when we have a root 3 for the characteristic equation

a polynomial of degree 1

Idea: Guess a particular solution $y_p(x)$ similar to $f(x)$

Should we guess $y_p(x) = \cancel{\underbrace{e^{3x}}_{\text{same form}} \underbrace{(Ax+B)}_{\text{a polynomial of degree 1}}}$? No!

Problem: we would get a term $e^{3x}B$ which is already part of $y_c(x) = C_1e^{2x} + C_2e^{3x}$.

Fix: Make $y_p(x)$ "different enough" by multiplying by x^{1}

because $r=3$ is a root of multiplicity 1 of the characteristic equation.

$$\boxed{\text{Guess} \quad y_p(x) = e^{3x}\, x(Ax+B) = e^{3x}(Ax^2+Bx)}$$

Step 3 Compute as many derivatives as we need to substitute into [N]:

$$y_p(x) = e^{3x}(Ax^2+Bx)$$

$$y_p'(x) = 3e^{3x}(Ax^2+Bx) + e^{3x}(2Ax+B) = e^{3x}[3Ax^2+(3B+2A)x+B]$$

$$y_p''(x) = 3e^{3x}[3Ax^2+(3B+2A)x+B] + e^{3x}[6Ax+(3B+2A)]$$

$$= e^{3x}[9Ax^2+(9B+12A)x+6B+2A]$$

Step 4 Substitute into the nonhomogeneous ODE [N]:

$$y''-5y'+6y = 2xe^{3x}$$

$$e^{3x}[9Ax^2+(9B+12A)x+6B+2A]$$

$$-5e^{3x}[3Ax^2+(3B+2A)x+B]+6e^{3x}(Ax^2+Bx) = 2xe^{3x}$$

$$e^{3x}[(9A-15A+6A)x^2+(9B+12A-15B-10A+6B)x+(6B+2A-5B)] = 2xe^{3x}$$

$$e^{3x}[0\quad x^2+\quad 2A\ x \qquad + B+2A] = 2xe^{3x}$$

Step 5 Solve for the coefficients A, B so that LHS and RHS are equal

$$2Ax+B+2A = 2x+0$$

$$2A = 2 \Rightarrow \boxed{A=1}$$

$$B+2A=0 \qquad B+2(1)=0 \Rightarrow \boxed{B=-2}$$

Step 6 Then a particular solution is $y_p(x) = e^{3x}(1x^2-2x)$

General solution of the nonhomogeneous ODE [N] is

$$y(x) = y_c(x) + y_p(x)$$

$$\boxed{y(x) = C_1e^{2x}+C_2e^{3x}+e^{3x}(x^2-2x), \quad C_1, C_2 \in \mathbb{R}}$$

Ex 2 ODE $y''' - y'' + y' - y = e^{2x}(x+1)$ [N]

Step 1 Find general solution of the homogeneous ODE

$$y''' - y'' + y' - y = 0 \qquad [H]$$

using Sec 3.3 method

Characteristic equation is

$$r^3 - r^2 + r - 1 = 0 \qquad [C]$$
$$(r-1)(r^2+1) = 0$$

$r=1$ is a root w/ multiplicity 1

$r = 0 \pm i$ are conjugate roots, each with multiplicity 1

A general solution of [H] is

$$y_c(x) = C_1 e^{1x} + e^{0x}\left[C_2 \cos(1x) + C_3 \sin(1x)\right]$$

$$y_c(x) = C_1 e^{x} + C_2 \cos(x) + C_3 \sin(x), \qquad C_1, C_2, C_3 \in \mathbb{R}$$

Step 2 Guess a particular solution $y_p(x)$ of the nonhomogeneous ODE [N]:

Because $f(x) = e^{2x}\underbrace{(x+1)}_{\text{polynomial of degree 1}}$

- $r=2$ is not a root of the characteristic equation [C]
- You can also think of $r=2$ as a root w/ multiplicity 0

guess $y_p(x) = e^{2x}\underbrace{(Ax+B)}_{\text{a polynomial of deg 1}}$

Check that 2 is not a root of the characteristic equation

Alternatively, check that e^{2x} is not already part of $y_c(x)$

Alternatively, guess $y_p(x) = x^0 e^{2x}(Ax+B)$

0 because $r=2$ is a root w/ multiplicity 0 of [C]

Step 3 Compute as many derivatives of $y_p(x)$ as we need for [N]

$$y_p(x) = e^{2x}(Ax+B)$$

$$y_p'(x) = 2e^{2x}(Ax+B) + e^{2x}(A) = e^{2x}(2Ax + 2B + A)$$

$$y_p''(x) = 2e^{2x}(2Ax+2B+A) + e^{2x}(2A) = e^{2x}(4Ax + 4B + 4A)$$

$$y_p'''(x) = 2e^{2x}(4Ax+4B+4A) + e^{2x}(4A) = e^{2x}(8Ax + 8B + 12A)$$

Step 4 Substitute y_p and its derivatives into the nonhomogeneous ODE [N]:

$$y''' - y'' + y' - y = e^{2x}(x+1)$$

$$e^{2x}(8Ax+8B+12A) - e^{2x}(4Ax+4B+4A)$$

$$+ e^{2x}(2Ax+2B+A) - e^{2x}(Ax+B) = e^{2x}(x+1)$$

$$e^{2x}\left[(8A-4A+2A-A)x + (8B-4B+2B-B) + (12A-4A+A)\right] = e^{2x}(x+1)$$

$$e^{2x}\left(5Ax + 5B + 9A\right) = e^{2x}(x+1)$$

Step 5 Find the coefficients so that LHS and RHS match

$$5Ax = 1x \Rightarrow 5A = 1$$

$$5B + 9A = 1$$

Solve for A, B: $\boxed{A = \frac{1}{5}}$

$$5B + 9\left(\frac{1}{5}\right) = 1$$

$$5B = 1 - \frac{9}{5} = -\frac{4}{5}$$

$$B = \boxed{-\frac{4}{25}}$$

Step 6 Then a particular solution is $y_p(x) = e^{2x}\left(\frac{1}{5}x - \frac{4}{25}\right)$

General solution of the nonhomogeneous ODE [N] is

$$y(x) = y_c(x) + y_p(x)$$

$$\boxed{y(x) = c_1 e^{2x} + c_2\cos(x) + c_3\sin(x) + e^{2x}\left(\frac{1}{5}x - \frac{4}{25}\right), \quad c_1, c_2, c_3 \in \mathbb{R}}$$

Fact 2

If

$y_{p_1}(x)$ is a solution of $a_n y^{(n)} + \dots + a_2 y'' + a_1 y' + a_0 y = f(x)$

and

$y_{p_2}(x)$ is a solution of $a_n y^{(n)} + \dots + a_2 y'' + a_1 y' + a_0 y = g(x)$,

then

$y_p(x) = y_{p_1}(x) + y_{p_2}(x)$ is a solution of

$$a_n y^{(n)} + \dots + a_2 y'' + a_1 y' + a_0 y = f(x) + g(x)$$

Ex 3 Solve the IVP $\underbrace{y^{(3)} + y' = 2 - \sin(x)}_{[N]}$; $y(0)=0$, $y'(0)=1$, $y''(0)=0$.

Step 1 Find general solution of the homogeneous ODE

$$y^{(3)} + y' = 0 \qquad [H]$$

The characteristic equation is

$$r^3 + r = 0$$
$$r(r^2+1) = 0$$

$r=0$ is a root w/ multiplicity 1

$r^2+1=0$
$r = 0 \pm i$ are roots w/ multiplicity 1

A general solution of [H] is

$$y_c(x) = c_1 e^{0x} + e^{0x}\left[c_2\cos(1x) + c_3\sin(1x)\right]$$

$$\boxed{y_c(x) = c_1 + c_2\cos(x) + c_3\sin(x)}$$

Step 2 The nonhomogeneous part is $f(x) = 2 - \sin x$

We now need to go through steps 3–6 for

$\underbrace{y^{(3)} + y' = 2}_{}$ [N. 1] AND $y^{(3)} + y' = \underbrace{-\sin(x)}_{f_2(x)}$ [N.2]

$\underbrace{2}_{f_1(x)}$

$f_1(x) = e^{0x}\, 2$

r=0 is a root of the characteristic equation w/ multiplicity 1

2 is a polynomial of degree 0

Guess $y_{p1}(x) = \underbrace{e^{0x}}_{\text{same}} x^{1} \underbrace{(A)}_{}$

multiplicity of r=0 is 1

polynomial of degree 0 w/ undetermined coefficient

$f_2(x) = e^{0x}\left[-\sin x\,(1) + \cos x\,(0)\right]$

polynomials of degree 0

The roots $0 \pm 1i$ are roots of the characteristic equation w/ multiplicity 1.

Guess

multiplicity of $\pm i$ is 1

$y_{p2}(x) = e^{0x} x^{1}\left[B\cos x + C\sin x\right]$

Step 3.1 Compute derivatives of $y_{p1}(x)$

$$y_{p1}(x) = Ax$$

$$y_{p1}'(x) = A$$

$$y_{p1}''(x) = y_{p1}'''(x) = 0$$

Step 3.2 Compute derivatives of $y_{p2}(x)$

$$y_{p2}(x) = x\left[B\cos x + C\sin x\right]$$

$$y_{p2}'(x) = B\cos x + C\sin x + x\left[-B\sin x + C\cos x\right]$$

$$y_{p2}''(x) = -B\sin x + C\cos x + \left[-B\sin x + C\cos x\right] + x\left[-B\cos x - C\sin x\right]$$

$$= -2B\sin x + 2C\cos x + x\left[-B\cos x - C\sin x\right]$$

$$y_{p2}'''(x) = -3B\cos x - 3C\sin x + x\left[B\sin x - C\cos x\right]$$

Step 4.1

Substitute into the nonhomogeneous ODE [N.1]

Not the original [N]

$$y^{(3)} + y' = 2$$

$$0 + A = 2$$

Step 4.2 Substitute into the nonhomogeneous ODE [N.2]

$$y^{(3)} + y' = -\sin(x)$$

- $-3B\cos x - 3C\sin x + x[B\sin x - C\cos x] + B\cos x + C\sin x + x[-B\sin x + C\cos x] = -\sin x$
- $-2B\cos x - 2C\sin x = -\sin x$

Step 5.1 $A = 2$

Step 5.2 $\left.\begin{array}{l}-2B=0\\-2C=-1\end{array}\right\} \Rightarrow \begin{array}{l}B=0\\C=\frac{1}{2}\end{array}$

Step 6.1 $y_{p1}(x) = 2x$

Step 6.2 $y_{p2}(x) = x\left[0\cos x + \frac{1}{2}\sin x\right]$

$$y_{p2}(x) = \frac{1}{2}x\sin x$$

Extra Step 7 $y_p(x) = y_{p_1}(x) + y_{p2}(x)$

$y_p(x) = 2x + \frac{1}{2}x\sin x$ is a solution to the original [N]

$$y^{(3)} + y' = 2 - \sin x$$

(by Fact 2)

So a general solution of [N] is $y(x) = y_c(x) + y_p(x)$

$$\boxed{y(x) = C_1 + C_2\cos(x) + C_3\sin(x) + 2x + \frac{1}{2}x\sin x, \quad C_1, C_2, C_3 \in \mathbb{R}}$$

Impose the initial conditions $y(0)=0,\ y'(0)=1,\ y''(0)=0$

$0 = y(0) = C_1 + C_2 1 + 0 + 0 + 0 \Rightarrow 0 = C_1 + C_2$

$y'(x) = -C_2\sin x + C_3\cos x + 2 + \frac{1}{2}(\sin x + x\cos x)$

$1 = y'(0) = 0 + C_3 + 2 \Rightarrow \boxed{C_3 = -1}$

$y''(x) = -C_2\cos x - C_3\sin x + \frac{1}{2}(\cos x + \cos x - x\sin x)$

$0 = y''(0) = -C_2 + \frac{1}{2}(1+1) \Rightarrow \boxed{C_2 = 1}$ so $\boxed{C_1 = -C_2 = -1}$

$$\boxed{\text{The solution to the IVP is } y(x) = -1 + \cos(x) - \sin(x) + 2x + \frac{1}{2}x\sin x}$$

Method of undetermined coefficients

(for finding a particular solution of a nonhomogeneous linear ODE)

Consider a nonhomogeneous linear ODE with constant coefficients

$$a_n y^{(n)} + \dots + a_2 y'' + a_1 y' + a_0 y = f(x) \qquad [N]$$

where $a_n, \dots, a_2, a_1, a_0 \in \mathbb{R}$ are constants and $a_n \neq 0$.

Step 1 Find roots of the characteristic equation

$$a_n r^n + \dots + a_2 r^2 + a_1 r + a_0 = 0$$

of the corresponding homogeneous ODE.

Step 2 Guess $y_p(x)$

Ex: $f(x) = e^{-6x}(3x^4 + x)$

$f(x) = e^{-6}$

$f(x) = (3x^4 + x)$

Case a : $f(x) = e^{cx}$ (a polynomial of degree m)

* If $r = c$ is a root with multiplicity s of the characteristic equation,

then guess $y_p(x) = e^{cx} x^s$ (a polynomial of degree m, with undetermined coefficients)

$$y_p(x) = e^{cx} x^s (A_m x^m + \dots + A_2 x^2 + A_1 x + A_0)$$

* If $r = c$ is not a root of the characteristic equation,

then think of $r = c$ as a root with multiplicity $s = 0$,

and guess $y_p(x) = e^{cx} x^0$ (a polynomial of degree m, with undetermined coefficients)

$$y_p(x) = e^{cx} (A_m x^m + \dots + A_2 x^2 + A_1 x + A_0)$$

Case b: $f(x) = e^{cx}\left[\cos(dx)\begin{pmatrix}\text{a polynomial of}\\ \text{degree } m_1\end{pmatrix} + \sin(dx)\begin{pmatrix}\text{a polynomial of}\\ \text{degree } m_2\end{pmatrix}\right]$

Ex $f(x) = 4\cos(2x)$ here $c=0$ and the 2nd polynomial is 0, $d=2$

$f(x) = e^{5x}\left[\cos(2x)(x^4+6) + \sin(2x)(3x)\right]$ $c=5$, $d=2$

$f(x) = e^{-x}\sin(6x)$ here $c=-1$, $d=6$, and the 1st polynomial is 0

* If $r = c \pm di$ are roots with multiplicity s of the characteristic equation,

then

set $m :=$ max of m_1 and m_2

and

guess

$$y_p(x) = e^{cx}x^s\left[\cos(dx)\begin{pmatrix}\text{a polynomial of deg } m,\\ \text{with undetermined coefficients}\end{pmatrix} + \sin(dx)\begin{pmatrix}\text{another polynomial of deg } m,\\ \text{with undetermined coefficients}\end{pmatrix}\right]$$

$$y_p(x) = e^{cx}x^s\left[\cos(dx)\left(A_m x^m + \dots + A_2x^2 + A_1x + A_0\right) + \sin(dx)\left(A_m x^m + \dots + A_2x^2 + A_1x + A\right)\right]$$

* If $r = c \pm di$ are not roots of the characteristic equation,

think of them as roots of multiplicity $s=0$, and

guess

$$y_p(x) = e^{cx}\left[\cos(dx)\left(A_m x^m + \dots + A_2x^2 + A_1x + A_0\right) + \sin(dx)\left(A_m x^m + \dots + A_2x^2 + A_1x + A\right)\right]$$

Remark: We have to include both sines and cosines
even if $f(x)$ only has one of them
because when we differentiate one we get the other

Step 3 Compute as many derivatives of $y_p(x)$ as we need:

$y_p'(x),\ y_p''(x),\ \dots,\ y_p^{(n)}(x)$ n is the order of the ODE

Step 4 Substitute $y_p(x)$ and its derivatives into the nonhomogeneous ODE

$$a_n y^{(n)} + \dots + a_2 y'' + a_1 y' + a_0 y = f(x) \qquad [N]$$

Step 5 Find the undetermined coefficients so that LHS and RHS match

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Step 6 Substitute these coefficients back into the guessed $y_p(x)$.

Extra Step 7

If $f(x)$ is a sum of terms of the form

Step 2 Case a and Step 2 case b,

then first apply the method for each term,

then take the sum of all the $y_p(x)$.

Ex If $f(x) = x^4 + 3x^2 - x + 10$ we only need to apply the method once because this fits into Step 2 case a: here $c=0$

If $f(x) = \underbrace{\cos(x)(x^4+3x^2)}_{\text{Case b}} + \underbrace{e^{-3x}}_{\text{Case a}}$, we need to apply the method twice, for each term