

Complex numbers

* $\mathbb{R} \stackrel{\text{def}}{=} \text{set of all real numbers}$

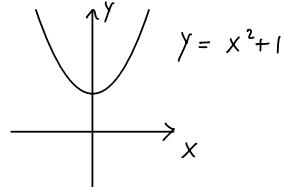
* Not every polynomial with coefficients in \mathbb{R} has roots in \mathbb{R} .

$$\text{Ex: } x^2 + 1 = 0$$

$$x^2 = -1$$

$$x = \pm \sqrt{-1}$$

not a real number



* We can extend \mathbb{R} to include all roots of polynomials

* Only need to add one new element $i \stackrel{\text{def}}{=} \sqrt{-1}$

Def The set of complex numbers \mathbb{C} consists of

elements of the form $z = a+bi$

$\begin{matrix} i \\ \uparrow \\ \text{both } a, b \text{ are real numbers} \end{matrix}$

Denote $\text{Re}(z) = a$ the real part of z

Denote $\text{Im}(z) = b$ the imaginary part of z

Note: Both $\text{Re}(z)$ and $\text{Im}(z)$ are real numbers.

• Every real number is a complex number whose imaginary part is 0.

Addition/ subtraction in \mathbb{C}

$$(a+bi)-(c+di) = a+bi-c-di = (a-c)+(b-d)i$$

Product in \mathbb{C}

$$(a+bi)(c+di) \stackrel{\text{foil}}{=} ac+adi+bci-bd = (ac-bd) + (ad+bc)i$$

x	a	bi
c	ac	bci
di	adi	bdi ² = -bd

Division in \mathbb{C}

If $a \neq 0$ or $b \neq 0$,

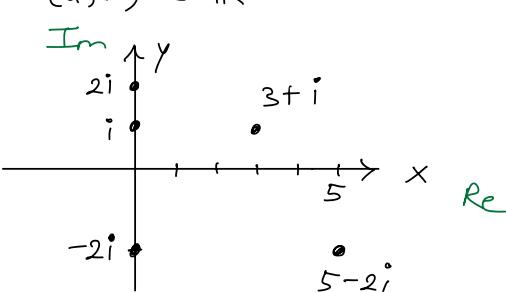
$$\frac{1}{a+bi} = \frac{1}{a+bi} \cdot \frac{a-bi}{a-bi} = \frac{a-bi}{a^2+b^2} = \frac{a}{a^2+b^2} - \frac{b}{a^2+b^2}i$$

Visualize \mathbb{C} as the xy -plane:

A complex number $z = a+bi \in \mathbb{C} \leftrightarrow (a,b) \in \mathbb{R}^2$

Ex

$$z = 5 - 2i$$



* The real numbers correspond to the x -axis.

* The complex numbers bi correspond to the y -axis.

Some vocab words

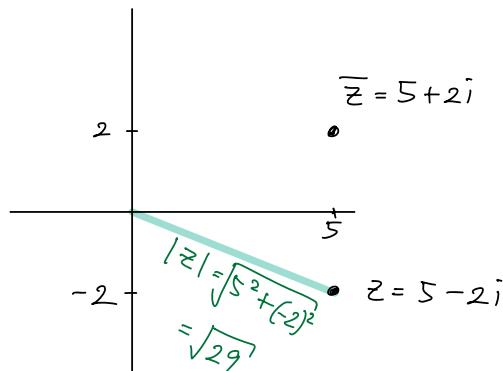
Let $z = a+bi$ be a complex number

* The modulus (or norm or absolute value) of z is $|z|$,
the distance $\sqrt{a^2+b^2}$ between z and the origin.

* The conjugate of z is $\bar{z} = a-bi$

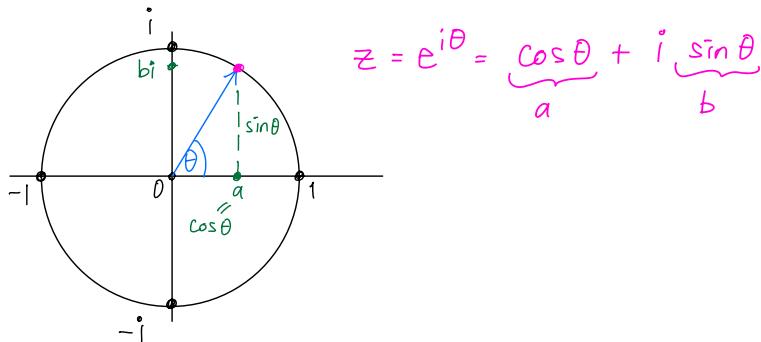
Note $z \cdot \bar{z} = (a+bi)(a-bi) = a^2 + b^2$

$$= |z|^2$$



Polar form of \mathbb{C}

Ex the set of complex numbers $z \in \mathbb{C}$ where $|z| = 1$ corresponds to the unit circle on the plane \mathbb{R}^2 :

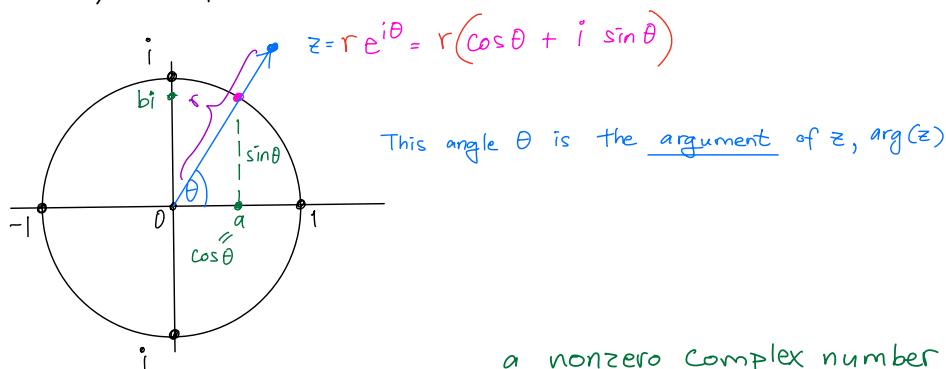


Observe: Every point on the unit circle is of the form $\cos \theta + i \sin \theta$.

Euler's formula: $e^{i\theta} = \cos \theta + i \sin \theta$

So define $e^{x+iy} = e^x e^{iy} = e^x [\cos y + i \sin y]$

To get other nonzero complex numbers, scale by a positive real number r , the modulus of z :



Def An argument $\arg(z)$ of $\underline{z \in \mathbb{Z} \setminus \{0\}}$ is a number θ

such that $z = |z| (\cos \theta + i \sin \theta)$

How to convert between polar and Cartesian representations

From polar to Cartesian: If $z = \underbrace{r e^{i\theta}}_{\text{modulus } |z|}$ then $z = r(\cos \theta + i \sin \theta)$

$$= \underbrace{r \cos \theta}_{\text{Re}(z)} + i \underbrace{r \sin \theta}_{\text{Im}(z)}$$

From Cartesian to polar: If $z = a + bi$, then $r = |z| = \sqrt{a^2 + b^2}$

$$\theta = \arg(z) = \arccos\left(\frac{b}{|z|}\right) = \arcsin\left(\frac{a}{|z|}\right)$$

and $z = r e^{i\theta}$

Note Cartesian form is better for addition,
but polar form is better for other arithmetic operations

* $(2 e^{\frac{\pi}{4}i})(5 e^{\frac{\pi}{3}i}) = (2)(5) e^{(\frac{\pi}{4} + \frac{\pi}{3})i} = 20 e^{\frac{7\pi}{12}i}$

* $\frac{2 e^{\frac{\pi}{4}i}}{5 e^{\frac{\pi}{3}i}} = \frac{2}{5} e^{(\frac{\pi}{4} - \frac{\pi}{3})i} = \frac{2}{5} e^{-\frac{\pi}{12}i}$

* If $z = r e^{i\theta}$ then $z^n = r^n e^{in\theta} = r^n (\cos(n\theta) + i \sin(n\theta))$

$$(2 e^{\frac{\pi}{4}i})^{10} = 2^{10} e^{\frac{\pi}{4} \cdot 10i} = 2^{10} \left(\cos\left(\frac{10\pi}{4}\right) + i \sin\left(\frac{10\pi}{4}\right) \right)$$

* Write $\sqrt{4i}$ in polar form $r e^{i\theta}$

Let $z = 4i = 4 e^{i\frac{\pi}{2}}$

$$\sqrt{4i} = \sqrt{z} = \sqrt{4} \sqrt{e^{i\frac{\pi}{2}}} = \pm 2 e^{i\frac{\pi}{4}}$$

because $e^{i\frac{\pi}{4}} e^{i\frac{\pi}{4}} = e^{i\frac{\pi}{2}}$

$$\cos\left(\frac{5\pi}{4}\right) + i \sin\left(\frac{5\pi}{4}\right) = e^{i\pi} e^{i\frac{\pi}{4}} = -e^{i\frac{\pi}{4}}$$

