$$\boxed{\text{Sec 3.3: Homogeneous equations w/ constant coefficients}}$$

→ we can solve them all

A small example 1

Consider a homogeneous 2nd-order linear ODE w/ constant coefficients

$$\underbrace{1\, y'' - 4 y' + 5 y}_{\text{constant}} = 0 \qquad [H]$$

Characteristic equation is (Think of y as the 0-th derivative of y)

$$r^2 - 4r + 5 = 0 \qquad [C]$$

Find roots of [C], using quadratic formula or "complete the square":

$$r^2 - 4r = -5$$

$$r^2 - 2.2r + 2^2 = -5 + 2^2$$

$$(r-2)^2 = -1$$

$$r - 2 = \pm\sqrt{-1} = \pm i$$

$$r = 2 \pm i \qquad a = 2 \qquad b = 1$$

The general solution of the ODE [H] is

$$y(x) = e^{ax}\left[c_1 \cos(bx) + c_2 \sin(bx)\right]$$

$$y(x) = \underbrace{e^{2x}\left[C_1 \cos(x) + C_2 \sin(x)\right]}_{\text{a linear combination of } e^{2x}\cos x \text{ and } e^{2x}\sin x}, \quad C_1, C_2 \in \mathbb{R}$$

Fundamental Thm of Algebra

A polynomial of degree n with coefficients in $\mathbb{C}$ has exactly n roots in $\mathbb{C}$ (counting multiplicities).

In particular, a polynomial of degree n with coefficients in $\mathbb{R}$ has exactly n roots in $\mathbb{C}$ (counting multiplicities).

Method for finding solution of homogeneous linear ODE w/ constant coefficients

Consider the n-th order ODE

$$a_n y^{(n)} + a_{n-1} y^{(n-1)} + \dots + a_2 y'' + a_1 y' + a_0 y = 0 \qquad [H]$$

where $a_n, a_{n-1}, \dots, a_2, a_1, a_0 \in \mathbb{R}$ are constants and $a_n \neq 0$

Step 1 Write the characteristic equation associated w/ [H]:

$$a_n r^n + a_{n-1} r^{n-1} + \dots + a_2 r^2 + a_1 r + a_0 = 0 \qquad [C]$$

It has exactly n complex roots, counted with multiplicities.

The roots can be:

- Real roots (with any multiplicity)
- Pairs of complex conjugate roots with the same multiplicity, that is, if $a+bi$ is a root with multiplicity m, then $a-bi$ is also a root with multiplicity m.

Ex $$(r-3)^4 (r+1)^5 \underbrace{(r^2+4)^3}_{=(r-2i)^3(r+2i)^3} = 0$$

has 15 roots, counted with multiplicity:

4 • 3 is a root with multiplicity 4.

5 • -1 is a root with multiplicity 5

3×2 • $2i$ is a root with multiplicity 3

Its conjugate $-2i$ is also a root with multiplicity 3

$+$ 15

$\boxed{\text{Step 2}}$ Each root with multiplicity m contributes m terms to the general solution of the ODE [H] :

* Each real root $r_1 \in \mathbb{R}$ of the characteristic equation [C] with multiplicity m contributes these m terms :

a linear combination of m "linearly independent" "fundamental" solutions

$$\overbrace{e^{r_1 x}, \quad x e^{r_1 x}, \quad x^2 e^{r_1 x}, \quad \ldots, \quad x^{m-1} e^{r_1 x}}$$

$$c_1 e^{r_1 x} + c_2 x e^{r_1 x} + c_3 x^2 e^{r_1 x} + \ldots + c_m x^{m-1} e^{r_1 x} =$$

$$(c_1 + c_2 x + c_3 x^2 + \ldots + c_m x^{m-1}) e^{r_1 x},$$

where $c_1, c_2, c_3, \ldots, c_m \in \mathbb{R}$.

<u>Ex</u> If 5 is a real root w/ multiplicity 3, it would contribute these 3 terms

$$(c_1 + c_2 x + c_3 x^2) e^{5x}, \qquad c_1, c_2, c_3 \in \mathbb{R}$$

* Each conjugate pair of complex roots $a+bi$ and $a-bi$ of the characteristic equation [C], each with multiplicity m, contributes these 2 m terms

a linear combination of 2m "linearly independent", "fundamental" solutions

$$e^{ax} \cos(bx), \quad e^{ax} x \cos(bx), \quad \ldots\ldots, \quad e^{ax} x^{m-1} \cos(bx),$$

$$e^{ax} \sin(bx), \quad e^{ax} x \sin(bx), \quad \ldots\ldots, \quad e^{ax} x^{m-1} \sin(bx),$$

$$A_1 e^{ax} \cos(bx) + A_2 e^{ax} x \cos(bx) + \ldots\ldots + A_m e^{ax} x^{m-1} \cos(bx)$$

$$+ B_1 e^{ax} \sin(bx) + B_2 e^{ax} x \sin(bx) + \ldots\ldots + B_m e^{ax} x^{m-1} \sin(bx),$$

$$A_1, \ldots, A_m, B_1, \ldots, B_m \in \mathbb{R}$$

Ex 2 Find general solution of ODE

$$9y^{(5)} - 6y^{(4)} + y^{(3)} = 0$$

5th-order

Ans

Step 1 Characteristic equation is

$$9r^5 - 6r^4 + r^3 = 0$$

$$r^3(9r^2 - 6r + 1) = 0$$

$$9r^2 - 6r + 1 = 0$$

$$(3r-1)^2 = 0$$

$$3r = 1$$

$r_2 = \frac{1}{3}$ is a root w/ multiplicity 2

$$r^3 = 0$$

$r_1 = 0$ is a root w/ multiplicity 3

Step 2 (A) The general solution of the ODE is

$$y(x) = (C_1 + C_2 x + C_3 x^2)\, e^{0x} + (C_4 + C_5 x)\, e^{\frac{1}{3}x}$$

$$= C_1 + C_2 x + C_3 x^2 + (C_4 + C_5 x)\, e^{\frac{1}{3}x}, \quad \underbrace{C_1, C_2, C_3, C_4, C_5 \in \mathbb{R}}$$

Sanity check: five constants because the ODE has order 5

Sanity check: The general solution should have 5 terms because the ODE has order 5.

The 5 functions should all look "different enough"

$1, x, x^2, e^{\frac{1}{3}x}, xe^{\frac{1}{3}x}$ are "different enough"

Ex 3 Find an ODE for which a general solution is

$$y(x) = C_1 + C_2 x + C_3 x^2 + C_4 x^3 + C_5 x^4$$
$$+ (C_6 + C_7 x + C_8 x^2) \cos(2x)$$
$$+ (C_9 + C_{10} x + C_{11} x^2) \sin(2x)$$

Eleven terms

Ans The form for the general solution fits a homogeneous 11-th order linear ODE w/ constant coefficients.

These five terms tell us we need a root $r_1 = 0$ w/ multiplicity 5

$$y(x) = (C_1 + C_2 x + C_3 x^2 + C_4 x^3 + C_5 x^4) e^{0x}$$
$$+ (C_6 + C_7 x + C_8 x^2) e^{0x} \cos(2x)$$
$$+ \underbrace{(C_9 + C_{10} x + C_{11} x^2)} e^{0x} \sin(2x)$$

$e^{0x} \cos(2x)$ and $e^{0x} \sin(2x)$ tell us we need a root $0 \pm 2i$

The three terms tell us the multiplicity of $0+2i$ and $0-2i$ is three each

A characteristic equation we want is

$$r^5 (r - (0+2i))^3 (r - (0-2i))^3 = 0$$
$$r^5 ((r - 2i)(r + 2i))^3 = 0$$
$$r^5 (r^2 + 4)^3 = 0$$

multiply out

$$r^5 (r^4 + 8r^2 + 16)(r^2 + 4)$$
$$r^5 (r^6 + 4r^4 + 8r^4 + 32r^2 + 16r^2 + 64) = 0$$
$$r^5 (r^6 + 12r^4 + 48r^2 + 64) = 0$$
$$r^{11} + 12r^9 + 48r^7 + 64r^5 = 0$$

Sanity check: The characteristic polynomial has degree $11 = 5 + 2 \cdot 3$

The ODE corresponding to this characteristic equation is

$$\boxed{y^{(11)} + 12 y^{(9)} + 48 y^{(7)} + 64 y^{(5)} = 0}$$

Ex 4 Solve $y^{(3)} + y' - 10y = 0$ Hint: $2^3 + 2 - 10 = 0$

Ans The characteristic equation is

$$r^3 + r - 10 = 0$$

Since $2^3 + 2 - 10 = 0$, 2 is a root,

so $(r-2)$ is a factor of $r^3 + r - 10$.

Perform long division to compute $\frac{r^3 + r - 10}{r-2}$:

$$\begin{array}{r|l} & r^2 + 2r + 5 \\ \hline r-2 & r^3 \qquad + r - 10 \\ & r^3 - 2r^2 \\ \hline & \quad 2r^2 + r - 10 \\ & \quad 2r^2 - 4r \\ \hline & \qquad 5r - 10 \\ & \qquad 5r - 10 \\ \hline & \qquad\quad 0 \end{array}$$

So $\frac{r^3 + r - 10}{r-2} = r^2 + 2r + 5$

So $r^3 + r - 10 = (r-2)(r^2 + 2r + 5)$

So $r - 2 = 0$ and $r^2 + 2r + 5 = 0$

One real root 2 w/ multiplicity 1

$$r^2 + 2r = -5$$
$$r^2 + 2r + 1 = -5 + 1$$
$$(r+1)^2 = -4$$
$$r + 1 = \pm\sqrt{-4} = \pm 2i$$
$$r = -1 \pm 2i$$

conjugate pair of complex roots, each with multiplicity 1

$$\boxed{\begin{array}{l}\text{General solution is} \\ y(x) = C e^{2x} + e^{-x}\left[A\cos(2x) + B\sin(2x)\right], \quad C, A, B \in \mathbb{R}.\end{array}}$$