Sec 3.1 2nd-order linear ODE part b

Recall vocabulary words:

* Two functions f and g are <u>linearly dependent</u> if $\frac{f}{g} = (\text{a constant})$ or $\frac{g}{f} = (\text{a constant})$

* f and g are <u>linearly independent</u> if they are not linearly dependent, that is, if $\frac{f}{g}$ is not a constant and $\frac{g}{f}$ is not a constant.

<u>Ex</u>

$f = e^x \qquad g = e^{-2x}$

$\frac{f}{g} = \frac{e^x}{e^{-2x}} = e^{x+2x} = e^{3x}$ is not a constant

So e^x and e^{-2x} are <u>linearly independent</u>

e^x and e^{x+2} are linearly dependent because $e^{x+2} = \underbrace{e^2}_{\text{a number}} e^x$

Another way to test for linear independence

We compute the <u>Wroskian</u> $W(f,g)$

(more powerful when dealing w/ linear ODEs of order 3 or higher)

<u>Def</u> The Wronskian of two functions $f(x)$ and $g(x)$ is the "determinant"

$$W(f,g) = \begin{vmatrix} f & g \\ f' & g' \end{vmatrix} = fg' - f'g$$

Theorem (Wronskian and linear independence)

Let $f(x)$ and $g(x)$ be two differentiable functions on an open interval I.

If $W(f,g) \neq 0$ for some x in I, → meaning $W(f,g)$ is not the constant zero function

then $f(x)$ and $g(x)$ are linearly independent

Same Ex $f = e^x$ $g = e^{-2x}$ Compute the Wronskian

Ans $f' = e^x$ $g' = -2e^{-2x}$

$$W(f,g) = \begin{vmatrix} f & g \\ f' & g' \end{vmatrix} = fg' - f'g = e^x(-2e^{-2x}) - e^x(e^{-2x})$$

$$= -2e^{-x} - e^{-x}$$

$$= -3e^{-x}$$

$-3e^{-x}$ is not the zero function

So $f(x) = e^x$ and $g(x) = e^{-2x}$ are linearly independent.

Thm (General solutions of homogeneous equations)

Let $y_1(x)$ and $y_2(x)$ be two linearly independent solutions of the homogeneous equation

$$y'' + p(x)\,y' + q(x)\,y = 0,$$

with $p(x)$ and $q(x)$ continuous on the open interval I.

Then any solution $y(x)$ can be written in the form

$$y(x) = c_1\, y_1(x) + c_2\, y_2(x) \quad \text{for all } x \in I.$$

for some numbers

Rem We say $y(x)$ is a "linear combination" of $y_1(x)$ and $y_2(x)$.

Q: Can we always find two linearly independent solutions of

$$y'' + p(x)\, y' + q(x)\, y = 0 \quad ?$$

A: Yes! By the Existence & Uniqueness Thm,

we can, on some interval I where $p(x)$ and q are continuous.

<u>How?</u>

By the thm, there is a unique solution $y_1(x)$ such that

$y_1(a) = 1$, $\quad y_1'(a) = 0$ for some $a \in I$

and there is a unique solution $y_2(x)$ such that

$y_2(a) = 0$, $\quad y_2'(a) = 1$ for some $a \in I$

<u>How do we know these $y_1(x)$ and $y_2(x)$ are linearly independent?</u>

If they were lin. dependent, then

$y_1 = k\, y_2$ for some k

(This would imply $1 = y_1(a) = k\, y_2(a) = 0$ which is impossible)

or $\quad y_2 = k\, y_1$ for some k

(implying $1 = y_2'(a) = k\, y_1'(a) = 0$ which is impossible),

so they are <u>not</u> lin. dependent.

Summary so far: We know that two linearly independent solutions of a homogeneous 2nd-order linear ODE exist, but we don't know how to compute them. This might be difficult.

Ex ODE $y'' - 4y = 0$

⌂ Check that $y_1(x) = e^{2x}$ and $y_2(x) = e^{-2x}$ are solutions to the ODE.

⌂ Check that they are linearly independent.

Then every solution of $y'' - 4y = 0$ can be written as

$$y(x) = c_1 e^{2x} + c_2 e^{-2x} \quad \text{for } c_1, c_2 \in \mathbb{R}$$

So this is a general solution

But consider $y_3(x) = \cosh(2x)$ $\qquad y_4(x) = \sinh(2x)$

$y_3'(x) = 2\sinh(2x)$ $\qquad y_4'(x) = \cosh(2x)$

$y_3''(x) = 4\cosh(2x)$ $\qquad y_4''(x) = \sinh(2x)$

$$y_3''(x) - 4y_3(x) = 0 \quad \text{and} \quad y_4''(x) - 4y_4(x) = 0$$

So $y_3(x)$ and $y_4(x)$ are also solutions to the ODE.

They are also linearly independent: $\dfrac{y_4(x)}{y_3(x)} = \dfrac{\sinh(2x)}{\cosh(2x)} = \tanh(2x)$ not a constant function

Then every solution of $y'' - 4y = 0$ can be written as

$$y(x) = c_1 \cosh(2x) + c_2 \sinh(2x) \quad \text{for } c_1, c_2 \in \mathbb{R}$$

So this is another general solution

Rem $\cosh(2x) \overset{\text{def}}{=} \frac{1}{2}e^{2x} + \frac{1}{2}e^{-2x}$ $\qquad \sinh(2x) \overset{\text{def}}{=} \frac{1}{2}e^{2x} - \frac{1}{2}e^{-2x}$

Upshot: We may find different pairs of linearly independent solutions of

$$y'' + p(x)y' + q(x) = 0.$$

Hence we have multiple ways of giving a general solution.

Thm (General solutions of nonhomogeneous linear equations)
Thm 5 in Sec 3.2

Every solution $y(x)$ on an open interval I of the nonhomogeneous 2nd-order linear ODE

$$y'' + p(x)\, y' + q(x)\, y = f(x) \qquad [*]$$

can be written as

$$y(x) = \underbrace{c_1 y_1(x) + c_2 y_2(x)}_{\text{a general solution of the homogeneous ODE}} + \underbrace{y_p(x)}_{\text{particular solution of } [*]}$$

where $y_1(x)$ and $y_2(x)$ are linearly independent solutions of the homogeneous equation

$$y'' + p(x)\, y' + q(x)\, y = 0$$

and $y_p(x)$ is any particular solution of $[*]$.

General solution of nonhomogeneous ODE = General solution of homogeneous ODE + Particular solution of nonhomogeneous ODE

(Will learn techniques for finding both)

Rem: Every concept & theorem in this section (for 2nd-order) also works for linear ODEs of order $n = 3$ and higher.

Techniques for linear 2nd-order homogeneous ODE w/ constant coefficients

Consider ODE $ay'' + by' + cy = 0$

numbers

Method for solving this ODE:

Step 1 Find the roots of the associated characteristic equation $ar^2 + br + c = 0$:

$$r = \frac{-b \pm \sqrt{b^2 - 4ac}}{2a}$$

Step 2

(a) If the characteristic equation has two distinct real roots r_1, r_2,

then $e^{r_1 x}$ and $e^{r_2 x}$ are two linearly independent solutions,

so a general solution is $y(x) = c_1 e^{r_1 x} + c_2 e^{r_2 x}$, $c_1, c_2 \in \mathbb{R}$

(b) If the characteristic equation has one real root r_1 w/ multiplicity 2

(ex $r^2 - 10r + 25 = 0$ has one real root 5 w/ multiplicity 2)
$(r-5)^2 = 0$

then $e^{r_1 x}$ and $xe^{r_1 x}$ are two linearly independent solutions,

so a general solution is
$$y(x) = c_1 e^{r_1 x} + c_2 x e^{r_1 x} = (c_1 + c_2 x) e^{r_1 x}, \quad c_1, c_2 \in \mathbb{R}$$

(c) If the characteristic equation has complex roots,

then ... STAY TUNED

Ex : ODE $y'' + 2y' = 0$

Step 1 The characteristic equation is $r^2 + 2r = 0$

$$r(r+2) = 0$$

Two distinct real roots 0 and -2

Step 2 A general solution is

$$y(x) = C_1 e^{0x} + C_2 e^{-2x}$$
$$= C_1 + C_2 e^{-2x}, \quad C_1, C_2 \in \mathbb{R}$$

Ex IVP $y'' + 2y' + y = 0$, $y(0) = 5$, $y'(0) = -3$

Step 1 The characteristic equation is $r^2 + 2r + 1 = 0$

$$(r+1)^2 = 0$$

Single real root -1 with multiplicity 2

Step 2 A general solution is

$$y(x) = C_1 e^{-x} + C_2 x e^{-x}, \quad C_1, C_2 \in \mathbb{R}$$

Step 3 Impose the initial conditions

$$5 = y(0) = C_1 e^0 + C_2 0 = C_1 \quad \text{so} \quad \boxed{C_1 = 5}$$

$$y'(x) = -C_1 e^{-x} + \underbrace{C_2 e^{-x} - C_2 x e^{-x}}_{\text{product rule}}$$

$$-3 = y'(0) = -C_1 e^0 + C_2 e^0 - C_2 0 = -5 + C_2 \quad \text{so} \quad \boxed{C_2 = 2}$$

The (particular) solution of the IVP is $\boxed{y(x) = 5e^{-x} + 2xe^{-x}}$.