

## Section 3.1

### Second-order linear equations

Def.: A second order linear equation is an equation of the form

$$A(x)y'' + B(x)y' + C(x)y = F(x)$$

Any function only in  $x$  (no  $y$ )

Example:  $e^x y'' + \cos(x) y' + (1+\sqrt{x}) y = \arctan(x)$   
is a second order linear equation.

$y'' + 3(y')^2 + 4y^3 = 0$  is not a  
linear equation.

Def.: The second -order linear equation

$$A(x)y'' + B(x)y' + C(x)y = F(x)$$

is called homogeneous if  $F(x) \equiv 0$  (it is the constant function equal to 0).

If  $F(x) \neq 0$  the equation is nonhomogeneous.

Then

$$A(x)y'' + B(x)y' + C(x)y = 0$$

is the homogeneous linear equation associated with the original one.

Example:  $x^2y'' + 2xy' + 3y = \cos(x)$

is nonhomogeneous, and its associated homogeneous equation is

$$x^2y'' + 2xy' + 3y = 0$$

Theorem (existence and uniqueness for second-order linear equations)

Suppose that the functions  $p, q$  and  $f$  are continuous on the open interval  $I$  containing the point  $a$ .

Then, given any two numbers  $b_0$  and  $b_1$ , the equation

$$y'' + p(x)y' + q(x)y = f(x)$$

has a unique solution on the entire interval  $I$  that satisfies the initial conditions

$$y(a) = b_0, \quad y'(a) = b_1.$$

Remark: If we have a linear equation of the form

$$A(x) y'' + B(x) y' + C(x) y = F(x),$$

before applying the theorem we will need to divide by  $A(x)$  to get an equation

$$y'' + \frac{B(x)}{A(x)} y' + \frac{C(x)}{A(x)} y = \frac{F(x)}{A(x)}$$

Then we apply the theorem on an interval  $I$  where  $\frac{B(x)}{A(x)}$ ,  $\frac{C(x)}{A(x)}$  and  $\frac{F(x)}{A(x)}$  are continuous; in particular  $A(x) \neq 0$  for every  $x \in I$ . Then the existence and uniqueness of the IVP over all the interval  $I$  (not just a subinterval) is guaranteed not only for the new equation (after dividing by  $A(x)$ ), but also for the original one.

recall that the existence and uniqueness theorem for first-order equations (not necessarily linear) only guarantees its conclusion on some interval contained (possibly smaller) in the rectangle where the function is "nice".

Remark: Since we have a second order equation, a general solution will involve two constants  $c_1, c_2$  and we need two initial conditions to determine them.

Usually we are given:

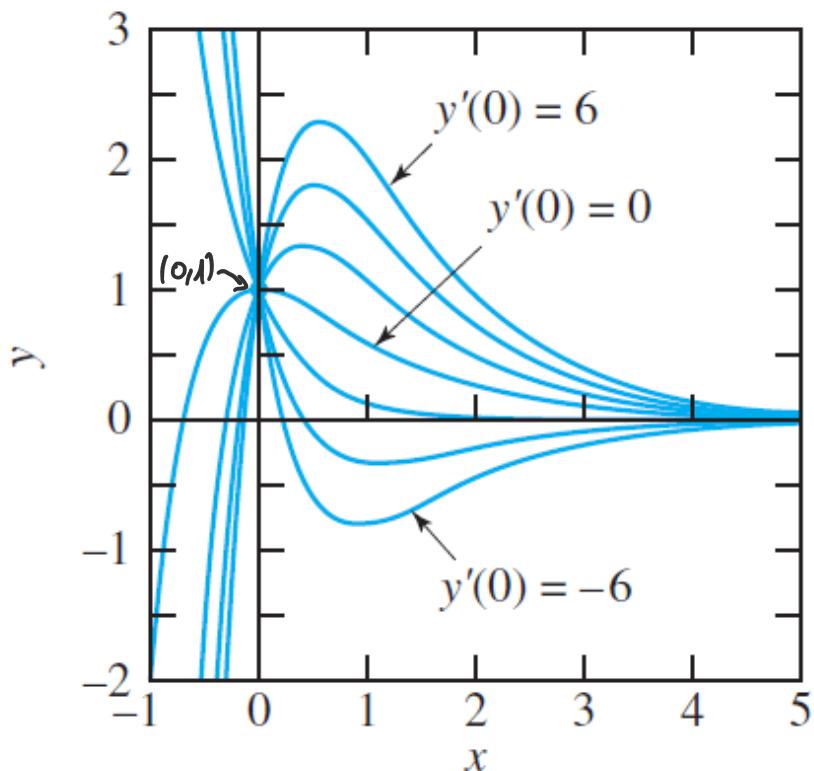
- The value of  $y(x)$  at some point  $x=a$ . That is, a point  $(a, y(a))$  which  $y(x)$  passes through.
- The value of  $y'(x)$  at the same point  $x=a$ . That is, the slope of the tangent line to  $y(x)$  at the point  $(a, y(a))$ .

Only one of those data is not enough to uniquely determine the solution.

Example: Consider the equation

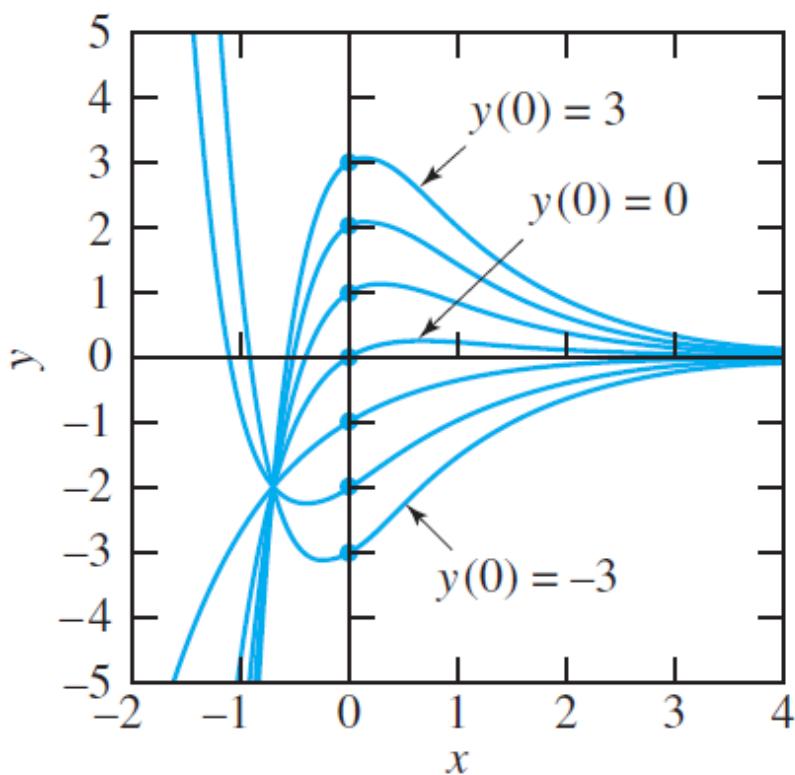
$$y'' + 3y' + 2y = 0$$

If we only impose the initial condition  $y(0)=1$ , we have infinitely many solutions going through  $(0,1)$ .



One solution for each value of the slope at  $x=0$ .

If we only impose the initial condition  $y'(0)=1$ , we have infinitely many solutions with slope 1 for  $x=0$ .



One solution through each point of the y-axis.

## Homogeneous second order linear equations

We will first study the solutions of homogeneous equations.

Nice (and important!) property:

Theorem (Principle of superposition for homogeneous equations)

Let  $y_1$  and  $y_2$  be two solutions of the homogeneous linear equation

$$A(x)y'' + B(x)y' + C(x)y = 0. \quad [1]$$

on the interval I. If  $c_1$  and  $c_2$  are constants, then the linear combination

$$y = c_1 y_1 + c_2 y_2$$

is also a solution of [1] on I.

Proof:  $y' = c_1 y_1' + c_2 y_2'$        $y'' = c_1 y_1'' + c_2 y_2''$

Then

$$\begin{aligned} A(x)y'' + B(x)y' + C(x)y &= \\ &= A(x)(c_1 y_1'' + c_2 y_2'') + B(x)(c_1 y_1' + c_2 y_2') + C(x)(c_1 y_1 + c_2 y_2) \\ &= c_1 \underbrace{(A(x)y_1'' + B(x)y_1' + C(x)y_1)}_0 + c_2 \underbrace{(A(x)y_2'' + B(x)y_2' + C(x)y_2)}_0 \\ &= 0. \end{aligned}$$

Example : Consider the equation

$$y'' + y = 0$$

We check that  $y_1(x) = \cos(x)$  and  $y_2(x) = \sin(x)$  are two solutions :

$$y_1'(x) = -\sin(x)$$

$$y_1''(x) = -\cos(x)$$

$$y_1''(x) + y_1(x) =$$

$$= -\cos(x) + \cos(x) = 0$$

$$y_2'(x) = \cos(x)$$

$$y_2''(x) = -\sin(x)$$

$$y_2''(x) + y_2(x) =$$

$$= -\sin(x) + \sin(x) = 0$$

Then for example  $y(x) = 3\cos(x) - 2\sin(x)$  is also a solution :

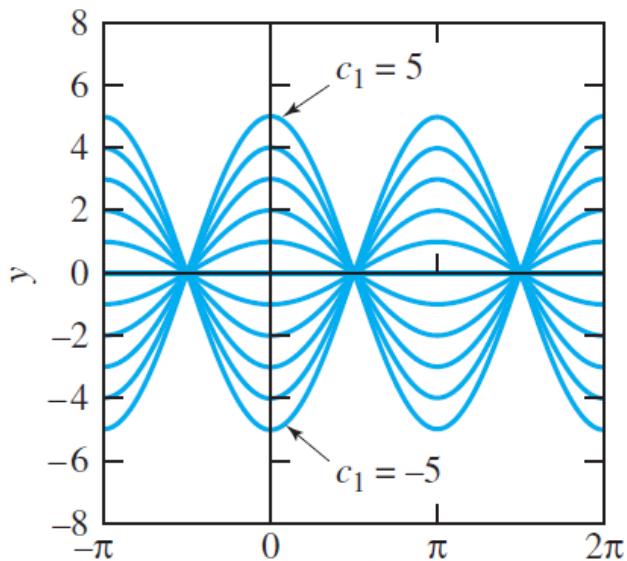
$$y'(x) = -3\sin(x) - 2\cos(x)$$

$$y''(x) = -3\cos(x) + 2\sin(x)$$

$$\begin{aligned} y''(x) + y(x) &= -3\cos(x) + 2\sin(x) + 3\cos(x) - 2\sin(x) \\ &= 0 \end{aligned}$$

This works for every value of  $c_1, c_2 \in \mathbb{R}$ ,

$y(x) = c_1 \cos(x) + c_2 \sin(x)$  is a solution of  $y'' + y = 0$ . ← a two-parameter family of solutions.

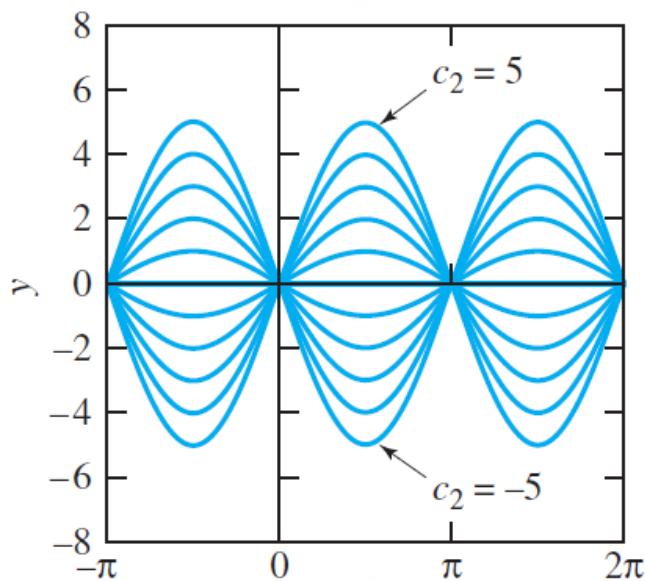


Solutions of the form

$$y(x) = c_1 \cos(x)$$

for different values  
of  $c_1 \in \mathbb{R}$ .

Here  $c_2 = 0$ .

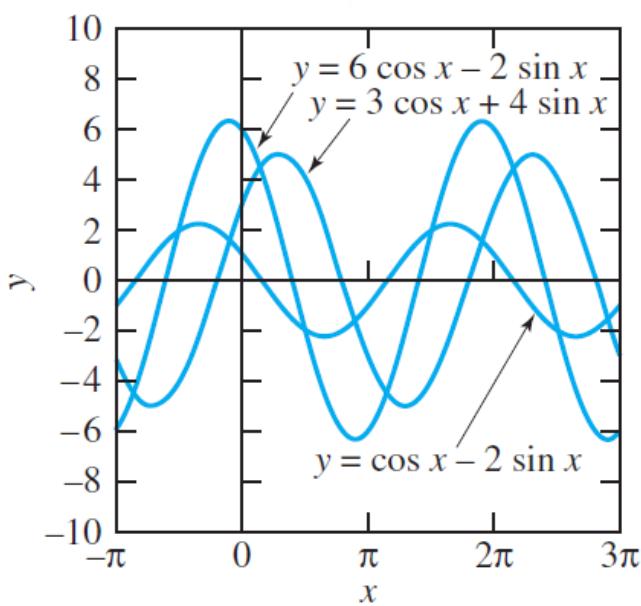


Solutions of the form

$$y(x) = c_2 \sin(x)$$

for different values  
of  $c_2 \in \mathbb{R}$ .

Here  $c_1 = 0$ .



Solutions of the form

$$y(x) = c_1 \cos(x) + c_2 \sin(x)$$

for  $c_1, c_2 \neq 0$ .

Example: For the same equation  $y''+y=0$ , we impose the initial conditions

$$y(0)=3, \quad y'(0)=-2.$$

We try to find a solution of the form

$$y(x)=c_1 \cos(x)+c_2 \sin(x)$$

satisfying those initial conditions.

$$y'(x) = -c_1 \sin(x) + c_2 \cos(x)$$

$$3 = y(0) = c_1 \cos(0) + c_2 \sin(0) = c_1$$

$$-2 = y'(0) = -c_1 \sin(0) + c_2 \cos(0) = c_2$$

Then  $y(x) = 3 \cos(x) - 2 \sin(x)$  satisfies  $y(0)=3$ ,  $y'(0)=-2$  and it is the unique function that does so, because of the existence and uniqueness theorem.

Question: Given two solutions  $y_1(x)$  and  $y_2(x)$  of

$$y'' + p(x)y' + q(x)y = 0,$$

We know that

$$c_1 y_1(x) + c_2 y_2(x)$$

is a two-parameter family of solutions.

Is it a general solution? That is, can all the possible solutions be written in this way for some  $c_1, c_2 \in \mathbb{R}$ ?

Answer: Yes, if  $y_1(x)$  and  $y_2(x)$  are "different enough".

The good notion of "different enough" is that they are linearly independent.

Def.: The  $n$  functions  $f_1, \dots, f_n$  are linearly dependent on the interval  $I$  if there exist  $c_1, \dots, c_n \in \mathbb{R}$  not all zero such that

$$c_1 f_1 + c_2 f_2 + \dots + c_n f_n = 0 \quad \text{on } I$$

$\uparrow c_1 f_1(x) + \dots + c_n f_n(x) = 0$   
for all  $x \in I$ .

$f_1, \dots, f_n$  are linearly independent on  $I$  if they are not linearly dependent:

$$c_1 f_1 + c_2 f_2 + \dots + c_n f_n = 0$$

only happens for  $c_1 = c_2 = \dots = c_n = 0$ .

For two functions:

$f_1, f_2$  are linearly dependent  $\iff$   $c_1 f_1 + c_2 f_2 = 0$   
with  $c_1 \neq 0$  or  $c_2 \neq 0$

$$\iff f_1 = \frac{-c_2}{c_1} f_2$$

or

$$f_2 = \frac{-c_1}{c_2} f_1$$

constant

$\Leftrightarrow$  one function is a constant multiple of the other.

$$\Leftrightarrow \frac{f_1}{f_2} = c$$

or  $\frac{f_2}{f_1} = c$

↑  
constant

$f_1, f_2$  are linearly independent  $\Leftrightarrow f_1, f_2$  are not linearly dependent

$$\Leftrightarrow \frac{f_1}{f_2} \neq c \text{ and } \frac{f_2}{f_1} \neq c$$

↑  
constant

$\Leftrightarrow$  neither function is a constant multiple of the other.

Example:  $f(x) = \sin(x)$ ,  $g(x) = \cos(x)$ .

$$\frac{f(x)}{g(x)} = \frac{\sin(x)}{\cos(x)} = \tan(x) \text{ which is not a constant function.}$$

Then  $\sin(x)$  and  $\cos(x)$  are linearly independent.

Example:  $f(x) = e^x$  and  $g(x) = e^{-2x}$

$$\frac{f(x)}{g(x)} = \frac{e^x}{e^{-2x}} = e^{3x} \text{ not a constant function}$$

Then  $e^x$  and  $e^{-2x}$  are linearly independent.

Example:  $f(x) = 0$ ,  $g(x)$  any function

Then  $f(x) = 0 \cdot g(x)$  and they are linearly dependent.

Example:  $f(x) = \sin(2x)$ ,  $g(x) = \sin(x)\cos(x)$

$$f(x) = \sin(2x) = 2 \sin(x)\cos(x) = 2g$$

Then they are linearly dependent.