Numerical methods for solving 1st order ODEs

Ref: Sec 2.4, 2.5, 2.6

* Many ODEs $\frac{dy}{dx} = f(x,y)$ are impossible to solve exactly

ex: $\frac{dy}{dx} = e^{-x^2}$

* Instead of finding the solution function $y(x)$, approximate its values $y_1 \approx y(x_1)$, $y_2 \approx y(x_2)$, $y_3 \approx y(x_3)$ ... at some values x_1, x_2, x_3, ...

The idea is similar to sketching an approximate solution curve in a slope field using only the short line segments

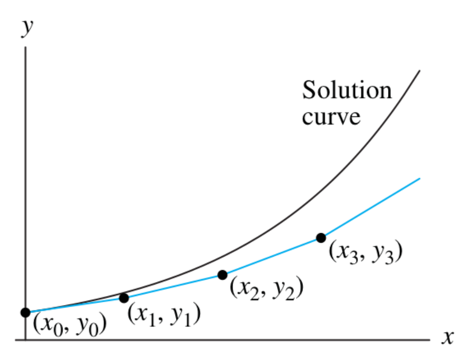


FIGURE 2.4.1. The first few steps in approximating a solution curve.

2.4 Numerical Approximation: Euler's Method

Euler's method

Consider an Initial Value Problem $\dfrac{dy}{dx} = f(x, y), \quad y(x_0) = y_0$

Choose a step size h

x_0, given $\qquad y_0$, given

$$x_1 = x_0 + h \qquad y_1 = y_0 + h \cdot f(x_0, y_0)$$
$$x_2 = x_1 + h \qquad y_2 = y_1 + h \cdot f(x_1, y_1)$$
$$x_3 = x_2 + h \qquad y_3 = y_2 + h \cdot f(x_2, y_2)$$
$$\vdots \qquad\qquad \vdots$$
$$x_{n+1} = x_n + h, \quad y_{n+1} = y_n + h \cdot \underbrace{f(x_n, y_n)}_{\text{slope at the previous point}}$$
$$\vdots \qquad\qquad \vdots$$

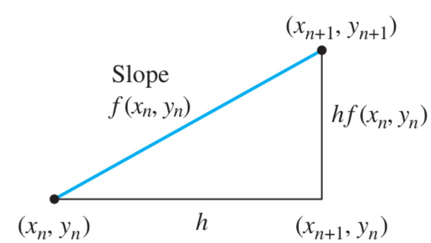


FIGURE 2.4.2. The step from (x_n, y_n) to (x_{n+1}, y_{n+1}).

Example

Apply Euler's method to approximate the solution of the initial value problem

$$\frac{dy}{dx} = x + y, \quad y(0) = 1$$

(The ODE is linear & we can solve it: $y(x) = 2e^x - x - 1$)

Choose step size $h = 0.2$ (like choosing density value in Geogebra slope field plotter)

$x_0 = 0$ (given) $\qquad y(x_0) = y_0 = 1$ (given)

$x_1 = x_0 + 0.2 = 0.2 \qquad y(x_1) \approx y_1 = y_0 + 0.2\, f(x_0, y_0) = 1 + 0.2 \cdot (0+1) = 1.2$

$x_2 = x_1 + 0.2 = 0.4 \qquad y(x_2) \approx y_2 = y_1 + 0.2\, f(x_1, y_1) = 1.2 + 0.2(0.2 + 1.2) = 1.48$

$x_3 = x_2 + 0.2 = 0.6 \qquad y(x_3) \approx y_3 = y_2 + 0.2\, f(x_2, y_2) = 1.48 + 0.2\,(0.4 + 1.48) = 1.856$

$x_4 = x_3 + 0.2 = 0.8 \qquad y(x_4) \approx y_4 = y_3 + 0.2\, f(x_3, y_3) = 1.856 + 0.2\,(0.6 + 1.856) = 2.3472$

$x_5 = x_4 + 0.2 = 1 \qquad y(x_5) \approx y_5 = y_4 + 0.2\, f(x_4, y_4) = 2.3472 + 0.2(0.8 + 2.3472) = 2.97664$

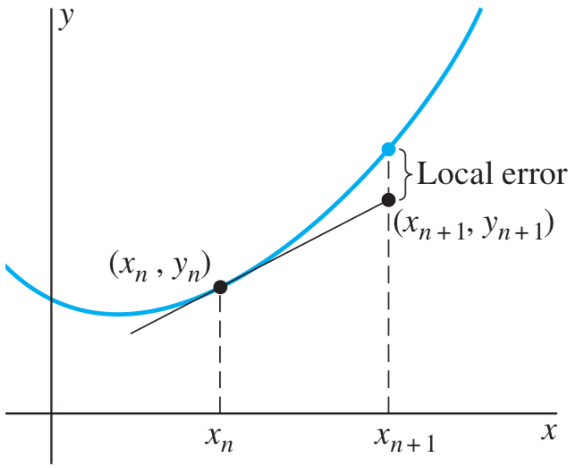
* Our approximation for $y(1)$ is $y_5 = 2.97664$
* But the actual value of $y(1)$ is $2e^1 - 1 - 1 \approx 3.4366$,

  so this approximation with $h = 0.2$ is not very good.
* With step size $h = 0.1$, we would get $y_{10} = 3.1875$
* With step size $h = 0.001$, we would get $y_{1000} = 3.4338$

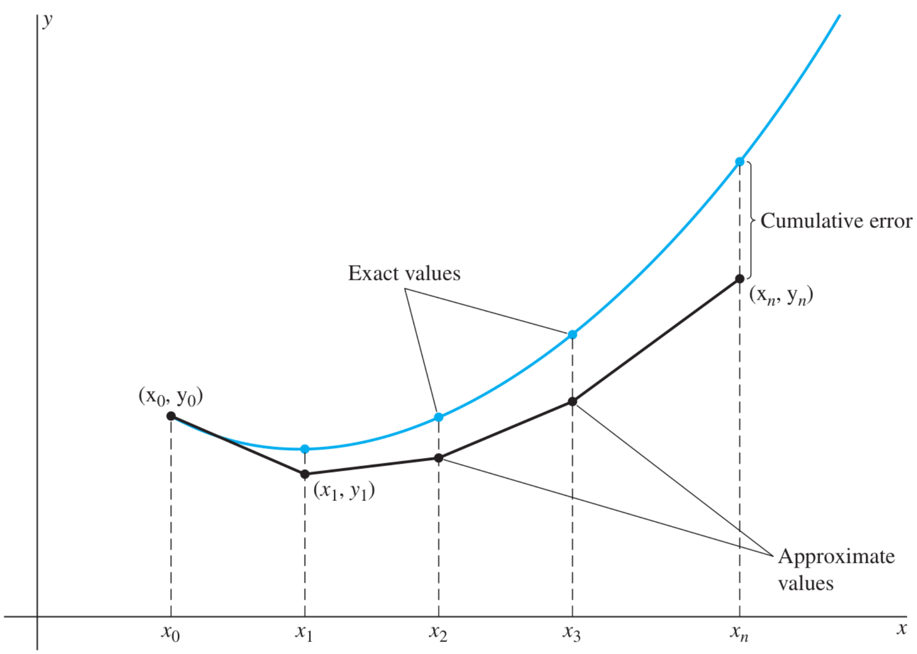
  much better, but this requires 1000 steps!

Local and Cumulative Errors

These make the approximation $y(x_n) \approx y_n$ unreliable when x_n is not sufficiently close to x_0



Local error is the error at each step: the slope $\frac{dy}{dx} = f(x,y)$ is not constant in the interval $[x_n, x_{n+1}]$, but we are using the slope at (x_n, y_n) to get to (x_{n+1}, y_{n+1}).



Cumulative error

After the first step, we are not starting at the actual point $(x_1, y(x_1))$ but at the approximation (x_1, y_1), and so on.

Roundoff error:

In the example, we could have written $y_3 = 1.86$ instead of $y_3 = 1.856$.

This would impact the computation of y_4, then y_5.

2.5 A Closer Look at the Euler Method

THEOREM 1 The Error in the Euler Method

Suppose that the initial value problem

$$\frac{dy}{dx} = f(x, y), \quad y(x_0) = y_0 \tag{1}$$

has a unique solution $y(x)$ on the closed interval $[a, b]$ with $a = x_0$, and assume that $y(x)$ has a continuous second derivative on $[a, b]$. (This would follow from the assumption that f, f_x, and f_y are all continuous for $a \leqq x \leqq b$ and $c \leqq y \leqq d$, where $c \leqq y(x) \leqq d$ for all x in $[a, b]$.) Then there exists a constant C such that the following is true: If the approximations $y_1, y_2, y_3, \ldots, y_k$ to the actual values $y(x_1), y(x_2), y(x_3), \ldots, y(x_k)$ at points of $[a, b]$ are computed using Euler's method with step size $h > 0$, then

$$|y_n - y(x_n)| \leqq Ch \tag{2}$$

for each $n = 1, 2, 3, \ldots, k$.

Upshot of Thm: * If the IVP $\frac{dy}{dx} = f(x,y)$, $y(x_0) = y_0$ is "nice"

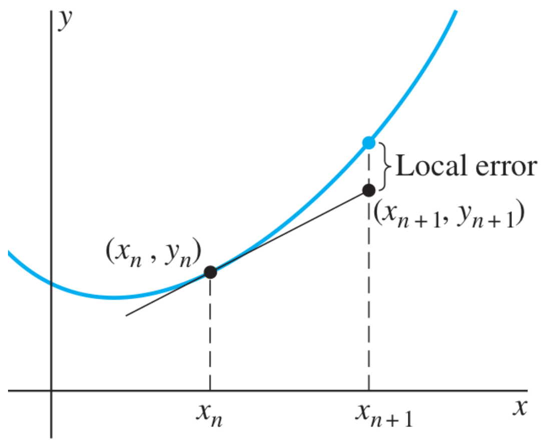
(unique solution, and f & its derivatives are continuous),

then there exists C such that $|y_n - y(x_n)| \leq Ch$

* That is, if we replace h by $\frac{h}{2}$, we expect the error to be divided by 2.

Euler's method

Given IVP $\frac{dy}{dx} = f(x,y)$, $y(x_0) = y_0$



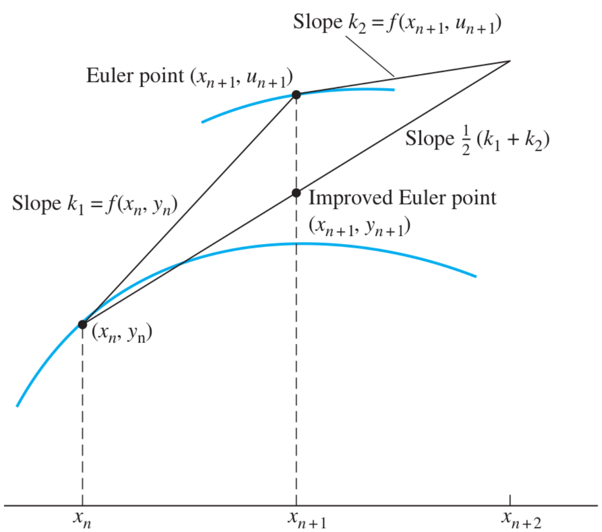
Approximating the slope of the solution curve $y(x)$ in the interval $[x_n, x_{n+1}]$ by $f(x_n, y_n)$, the slope at the starting point, gives big errors.

Improved Euler's method

Idea:

Instead of using the slope at the starting point, take the average between the slope at the starting point (x_n, y_n) and the slope at the predicted final point.

Improved Euler's method



Given: IVP $\frac{dy}{dx} = f(x, y), \quad y(x_0) = y_0$

* Fix a step size h

* Suppose we have computed (x_n, y_n).

$x_{n+1} = x_n + h$ (same as before)

To compute y_{n+1}, do:

$$
\begin{aligned}
k_1 &= f(x_n, y_n), \\
u_{n+1} &= y_n + h \cdot k_1, \\
k_2 &= f(x_{n+1}, u_{n+1}), \\
y_{n+1} &= y_n + h \cdot \underbrace{\tfrac{1}{2}(k_1 + k_2)}_{\text{average of slopes at the starting point and at "Euler point"}}
\end{aligned}
$$

u_{n+1} is the predicted y value using original Euler's method

average of slopes at the starting point and at "Euler point"

To perform the Improved Euler's method:

Start at (x_0, y_0) and perform the iterative formula until we get to the desired x value.

Cumulative error in Improved Euler's method:

* If the IVP $\frac{dy}{dx} = f(x, y), \quad y(x_0) = y_0$ is nice, then $|y(x_n) - y_n| \leq Ch^2$,

* If we replace h by $\frac{h}{2}$, we expect the error to be divided by 2^2.

$\boxed{\text{Same example}}$ IVP $\frac{dy}{dx} = x + y, \quad y(0) = 1$

(The ODE is linear & we can find the exact solution: $y(x) = 2e^x - x - 1$)

Choose step size $h = 0.2$

$$u_{n+1} = y_n + h \cdot (x_n + y_n),$$
$$y_{n+1} = y_n + h \cdot \tfrac{1}{2}\left[(x_n + y_n) + (x_{n+1} + u_{n+1})\right]$$

$x_0 = 0$ $\qquad y_0 = 1$

$x_1 = x_0 + 0.2 = 0.2$

$$u_1 = y_0 + 0.2\, f(x_0, y_0) = 1 + 0.2\,(0+1) = 1.2$$

$$y_1 = y_0 + 0.2\,\frac{f(x_0,y_0) + f(x_1,u_1)}{2} = 1 + 0.2\,\frac{(0+1)+(0.2+1.2)}{2} = 1.24$$

$x_2 = x_1 + 0.2 = 0.4$

$$u_2 = y_1 + 0.2\, f(x_1,y_1) = 1.24 + 0.2\,(0.2 + 1.24) = 1.528$$

$$y_2 = y_1 + 0.2\,\frac{f(x_1,y_1) + f(x_2,u_2)}{2} = y_1 + 0.2\,\frac{(0.2+1.24)+(0.4+1.528)}{2} = 1.5768$$

$x_3 = x_2 + 0.2 = 0.6$

$$u_3 = 1.5768 + 0.2\,(0.4 + 1.5768) = 1.97216$$

$$y_3 = 1.5768 + 0.2\,\frac{(0.4 + 1.5768) + (0.6 + 1.97216)}{2} = 2.031696$$

$x_4 = x_3 + 0.2 = 0.8$

$$u_4 = 2.5580352$$

$$y_4 = 2.63066912$$

$x_5 = x_4 + 0.2 = 1$

$$u_5 = 3.316802944$$

$$y_5 = 3.405416326 \approx y(x_5) = y(1) = \underbrace{2e^1 - 1 - 1}_{\text{actual value}} \quad (\text{close to } 3.4366)$$

The improved Euler's method gives better approximation than the original Euler's method with the same $h = 0.2$.

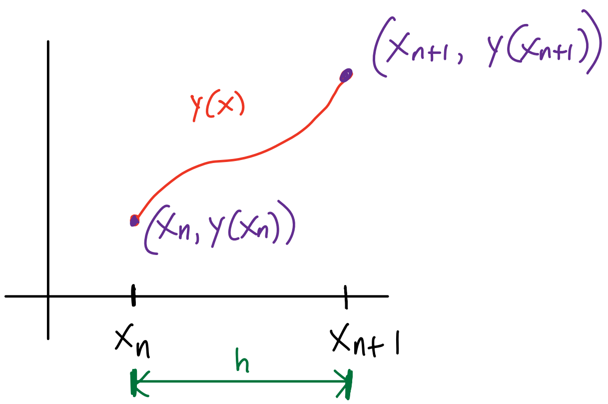
$y(1) \approx 2.97664$ using the original Euler's method.

Rem

Indeed, the improved Euler method with $h = 0.1$ is more accurate (in this example) than the original Euler method with $h = 0.005$. The latter requires 200 evaluations of the function $f(x, y)$, but the former requires only 20 such evaluations, so in this case the improved Euler method yields greater accuracy with only about one-tenth the work.

2.6 The Runge–Kutta Method

Idea: Instead using line segments, use integrals.



Given: IVP $\dfrac{dy}{dx} = f(x, y), \quad y(x_0) = y_0$

Fundamental Theorem of Calculus

$$y(x_{n+1}) - y(x_n) = \int_{x_n}^{x_{n+1}} y'(x)\,dx = \int_{x_n}^{x_n+h} y'(x)\,dx$$

Definite integrals $\int_a^b g(x)\,dx$ can be approximated numerically,

(ex: in Calc 2 or 3, use a partial sum of an infinite series)

Simpson's rule

$$\int_a^b g(x)\,dx \approx \frac{b-a}{6}\left[g(a) + 4g\left(\frac{a+b}{2}\right) + g(b)\right]$$

Applying Simpson's rule, we get

$$\underbrace{y(x_{n+1})}_{\substack{\approx \\ y_{n+1}}} - \underbrace{y(x_n)}_{\substack{\approx \\ y_n}} = \int_{x_n}^{x_{n+1}} y'(x)\,dx = \int_{x_n}^{x_n+h} y'(x)\,dx$$

$$\approx \frac{h}{6}\left[y'(x_n) + \underbrace{4y'}\left(x_n + \frac{h}{2}\right) + y'(x_{n+1})\right]$$

we split into two because we'll use different approximations

Hence we want to define y_{n+1} so that

$$y_{n+1} \approx y_n + \frac{h}{6}\left[\underbrace{y'(x_n)}_{\substack{\approx \\ k_1}} + \underbrace{2y'\left(x_n + \frac{h}{2}\right)}_{\substack{\approx \\ k_2}} + \underbrace{2y'\left(x_n + \frac{h}{2}\right)}_{\substack{\approx \\ k_3}} + \underbrace{y'(x_{n+1})}_{\substack{\approx \\ k_4}}\right]$$

We will compute the approximations k_1, k_2, k_3, k_4 as follows:

Fix a step h.

Starting from (x_n, y_n):

$x_{n+1} = x_n + h$ (like usual)

$$k_1 = f(x_n, y_n)$$ (Euler's method) slope at (x_n, y_n)

$$k_2 = f\left(x_n + \tfrac{1}{2}h, \underbrace{y_n + \tfrac{1}{2}hk_1}\right)$$

like Euler's method w/ step $\frac{h}{2}$: Start at (x_n, y_n), then move along line segment w/ slope $f(x_n,y_n)=k_1$

$$k_3 = f\left(x_n + \tfrac{1}{2}h, y_n + \tfrac{1}{2}hk_2\right)$$

like improved Euler's method w/ step $\frac{h}{2}$:
move along line segment w/ "improved" slope k_2.

$$k_4 = f(\underbrace{x_{n+1}}_{x_n+h}, y_n + hk_3)$$

Euler's method step w/ (improved) slope k_3

* Define $$y_{n+1} = y_n + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$

Rem Each of k_1, k_2, k_3, k_4 can be seen as an "improved" slope, and we take the average.

This is often called "RK 4"

Cumulative error in Runge-Kutta(4) method:

* If the IVP $\frac{dy}{dx} = f(x, y), \quad y(x_0) = y_0$ is nice, then $|y(x_n) - y_n| \leq Ch^4$,

* If we replace h by $\frac{h}{2}$, we expect the error to be divided by 2^4.

$\boxed{\text{Same example}}$ IVP $\dfrac{dy}{dx} = x + y, \quad y(0) = 1$

(The ODE is linear & we can find the exact solution: $y(x) = 2e^x - x - 1$)

Choose step size $h = 0.5$ (bigger than before)

$x_0 = 0$ $\qquad$ $y_0 = 1$

$x_1 = x_0 + 0.5 = 0.5$

$$k_1 = 0 + 1 = 1,$$
$$k_2 = (0 + 0.25) + (1 + (0.25) \cdot (1)) = 1.5,$$
$$k_3 = (0 + 0.25) + (1 + (0.25) \cdot (1.5)) = 1.625,$$
$$k_4 = (0.5) + (1 + (0.5) \cdot (1.625)) = 2.3125,$$
$$y_1 = 1 + \frac{0.5}{6}[1 + 2 \cdot (1.5) + 2 \cdot (1.625) + 2.3125] \approx 1.7969.$$

$x_2 = x_1 + 0.5 = 1$

$$k_1 = f(x_1, y_1) = 0.5 + 1.7969 = 2.2969$$
$$k_2 = f\left(x_1 + \frac{h}{2}, y_1 + \frac{h}{2} k_1\right)$$
$$k_3 = f\left(x_1 + \frac{h}{2}, y_1 + \frac{h}{2} k_2\right)$$
$$k_4 = f(x_2, y_1 + h k_3)$$
$$y_2 = y_1 + \frac{h}{6}(k_1 + 2k_2 + 2k_3 + k_4)$$
$$y_2 \approx 3.4347.$$

The actual value $y(x_2) = y(1)$ is close to 3.4366.

The RK4 method w/ step $h = 0.5$ is better than th original Euler's method w/ step 0.2 (2.97664) & Improved Euler's method w/ step 0.2 (3.4054).