Sec 1.5 Linear first-order equations

Application : mixture problems

Example 1

A 50 L tank initially contains 30 kg of salt dissolved in 40 L of water. A mixture containing 0.25 kg/L of salt flows into the tank at 3 L/sec & the mixture flows out of the tank at 2 L/sec.
How much salt does the tank contain when it is full?

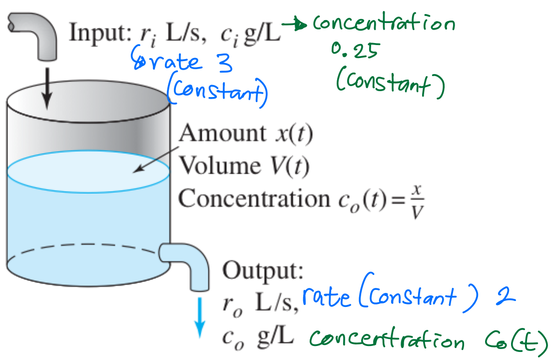

FIGURE 1.5.4. The single-tank mixture problem.

$V(t)$: volume of liquid (L)

$x(t)$: amout of salt (kg)

$x(0) = 30$

$$V(t) = \underbrace{40}_{\text{initial volume}} + (\underbrace{3}_{\text{rate of inflow}} - \underbrace{2}_{\text{rate of outflow}})\, t$$

$$\frac{dx}{dt} = \text{concentration of salt in incoming liquid} \cdot \text{inflow rate} - \text{concentration of salt in tank} \cdot \text{outflow rate}$$

$$\frac{dx}{dt} = 0.25 \, g/L \;\; 3 \, L/sec - \frac{x(t)\, g}{V(t)\, L} \;\; 2 \, L/sec$$

$$\frac{dx}{dt} = (0.25)\,3 - \frac{x(t)}{40+t}\,2$$

$$\frac{dx}{dt} + \underbrace{\frac{2}{40+t}}_{P(t)}\,x(t) = \underbrace{0.75}_{Q(t)}$$ is a linear 1st-order ODE.

We can solve it using the Integrating Factor method.

Step 1 Compute the integrating factor

$$\int P(t)\,dt = \int \frac{2}{40+t}\,dt = 2\ln|40+t| = 2\ln(40+t)$$

$40+t > 0$

$$e^{\int P(t)\,dt} = e^{2\ln(40+t)} = e^{\ln[(40+t)^2]} = (40+t)^2$$

Step 2 Multiply both sides by $(40+t)^2$

$$(40+t)^2\frac{dx}{dt} + (40+t)^2\frac{2}{40+t}\,x(t) = (40+t)^2\,0.75$$

$$\underbrace{(40+t)^2\frac{dx}{dt} + (40+t)\,2\,x(t)}_{\frac{d}{dx}\left[(40+t)^2 x(t)\right]} = (40+t)^2\,0.75$$

Step 3 Recognize LHS as a derivative

Step 4 Integrate both sides.

$$(40+t)^2 x(t) = \int 0.75 \, (40+t)^2 \, dt + C$$

$$= \frac{0.75 \, (40+t)^3}{3} + C$$

$$= 0.25 \, (40+t)^3 + C$$

Step 5 Solve for $x(t)$ to get explicit general solution

$$x(t) = 0.25 \, (40+t) + \frac{C}{(40+t)^2}$$

Step 6 Impose initial condition $x(0)=30$

$$30 = x(0) = 0.25 \, (40+0) + \frac{C}{(40+0)^2}$$

$$30 = 10 + \frac{C}{1600}$$

$$20 = \frac{C}{1600}$$

$$32\,000 = C$$

Then $x(t) = 0.25 \, (40+t) + \dfrac{32\,000}{(40+t)^2}$

is the particular solution to

$$\frac{dx}{dt} + \underbrace{\frac{2}{40+t}}_{P(t)} x(t) = \underbrace{0.75}_{Q(t)} \quad , \quad x(0) = 30$$

At what time is the tank full?

Set $V(t) = 50$ because the tank's capacity is 50 L

$40 + t = 50$

After $t = 10$ secs is when the tank is full

Amount of salt at 10 sec is

$$x(10) = 0.25(40 + 10) + \frac{32000}{50^2} \text{ kg}$$

$$\approx 25.3 \text{ kg}$$

Example 2

Note:
volume stays constat because all three flow rates are 5 gal/min

$V_1 = 100$ gal

$V_2 = 200$ gal

38. Consider the *cascade* of two tanks shown in Fig. 1.5.5, with $V_1 = 100$ (gal) and $V_2 = 200$ (gal) the volumes of brine in the two tanks. Each tank also initially contains 50 lb of salt. The three flow rates indicated in the figure are each 5 gal/min, with pure water flowing into tank 1. **(a)** Find the amount $x(t)$ of salt in tank 1 at time t. **(b)** Suppose that $y(t)$ is the amount of salt in tank 2 at time t. Show first that

$$\frac{dy}{dt} = \frac{5x}{100} - \frac{5y}{200},$$

and then solve for $y(t)$, using the function $x(t)$ found in part (a). **(c)** Finally, find the maximum amount of salt ever in tank 2.

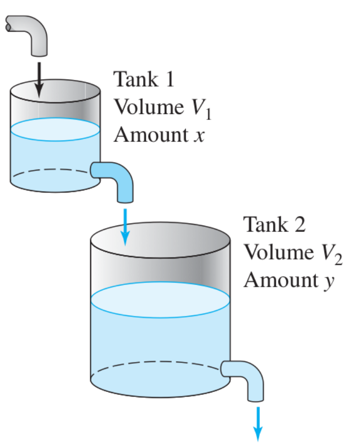

FIGURE 1.5.5. A cascade of two tanks.

a) Find $x(t)$: salt in tank 1 at minute t

$$\frac{dx}{dt}\frac{lb}{min} = \underset{\uparrow \text{pure water flows in}}{0} - \frac{x(t)\ lb}{100\ gal} \quad 5\ \frac{gal}{min}$$

The ODE is $\frac{dx}{dt} + \underbrace{\frac{5}{100}}_{P(t)=\frac{1}{20}} x(t) = \underbrace{0}_{Q(t)=0}$

which is a linear 1st-order ODE

Step 1: Integrating factor is $e^{\int \frac{1}{20} dt} = \boxed{e^{\frac{t}{20}}}$

Step 2: Multiply both sides of ODE by the integrating factor

$$e^{\frac{t}{20}} \frac{dx}{dt} + e^{\frac{t}{20}} \frac{1}{20} x(t) = 0$$

Step 3: Recognize the LHS as a derivative of a product

$$\frac{d}{dt}\left[e^{\frac{t}{20}} x(t)\right] = 0$$

Step 4: Integrate both sides

$$e^{\frac{t}{20}} x(t) = \int 0 \ dt + C$$
$$= C$$

Step 5: Solve for $x(t)$ to get explicit general solution

$$x(t) = C e^{-\frac{t}{20}}$$

Step 6: Find particular solution by imposing initial condition

$x(0) = 50$ because each tank initially contains 50 lb of salt

$$50 = x(0) = C e^0$$
$$50 = C$$

$\boxed{x(t) = 50 e^{-\frac{t}{20}}}$ is salt (in lbs) in tank 1 at minute t

(b) Let $y(t)$ be the amount of salt in tank 2 at minute t.

$$\frac{dy}{dt} = \underbrace{\frac{x(t)}{V_1}\frac{lb}{gal}\cdot 5\,gal/min}_{\text{salt in flow}} - \underbrace{\frac{y(t)}{V_2}\frac{lb}{gal}\cdot 5\,gal/min}_{\text{salt outflow}}$$

$$\frac{dy}{dt} = \frac{x(t)}{100}\,5 - \frac{y(t)}{200}\,5$$

$$\frac{dy}{dt} = 50\,e^{-\frac{t}{20}}\frac{5}{100} - y\,\frac{5}{200}$$

$$\frac{dy}{dt} + \underbrace{\frac{1}{40}}_{P(t)=\frac{1}{40}}y = \underbrace{\frac{5}{2}e^{-\frac{t}{20}}}_{Q(t)=\frac{5}{2}e^{-\frac{t}{20}}}$$ is a linear 1st-order ODE

Step 1 Integrating factor is $e^{\int \frac{1}{40}dt} = \boxed{e^{\frac{1}{40}t}}$

Step 2 Multiply both sides by the integrating factor

$$e^{\frac{1}{40}t}\frac{dy}{dt} + e^{\frac{1}{40}t}\frac{1}{40}y = e^{\frac{1}{40}t}\frac{5}{2}e^{-\frac{t}{20}}$$

Step 3 $$\frac{d}{dx}\left[e^{\frac{1}{40}t}y(t)\right] = \frac{5}{2}e^{-\frac{t}{40}}$$

Step 4 Integrate both sides

$$e^{\frac{1}{40}t}\, y(t) = \int \frac{5}{2} e^{-\frac{t}{40}}\, dt + C$$

$$= \frac{\frac{5}{2} e^{-\frac{t}{40}}}{-\frac{1}{40}} + C$$

$$= -40 \cdot \frac{5}{2} e^{-\frac{t}{40}} + C$$

$$= -100\, e^{-\frac{t}{40}} + C$$

Step 5 Get explicit general solution $y(t)$

$$y(t) = -100\, e^{-\frac{t}{40}} e^{-\frac{1}{40}t} + C e^{-\frac{1}{40}t}$$

$$y(t) = -100\, e^{-\frac{t}{20}} + C e^{-\frac{1}{40}t}$$

Step 6 Get particular solution by imposing initial condition $y(0) = 50$:

$$50 = y(0) = -100\, e^{0} + C e^{0}$$

$$50 = -100 + C$$

$$150 = C$$

So $$\boxed{y(t) = -100\, e^{-\frac{t}{20}} + 150\, e^{-\frac{1}{40}t}}$$ amount of salt (lbs) at minute t.

Sanity check: verify is 50 when $t = 0$

(c) Find the maximum amount of salt ever in tank 2.

Maximize $y(t)$ for t in $[0, \infty)$.

Find $\underline{\text{critical points}}$ t (where $y'(t)=0$ or $y'(t)$ doesn't exist):

$$y'(t) = -100\left(-\frac{1}{20}\right)e^{-\frac{t}{20}} + 150\left(-\frac{1}{40}\right)e^{-\frac{1}{40}t}$$

$$y'(t) = 5e^{-\frac{t}{20}} - \frac{15}{4}e^{-\frac{t}{40}}$$

Set $y'(t)=0$:

$$0 = 5e^{-\frac{t}{20}} - \frac{15}{4}e^{-\frac{t}{40}}$$

$$\frac{15}{4}e^{-\frac{t}{40}} = 5e^{-\frac{t}{20}}$$

$$\frac{3}{4}e^{-\frac{t}{40}}e^{\frac{t}{20}} = 1$$

$$e^{\frac{t}{40}} = \frac{4}{3}$$

$$\ln\left(e^{\frac{t}{40}}\right) = \ln\left(\frac{4}{3}\right)$$

$$\frac{t}{40} = \ln\left(\frac{4}{3}\right)$$

$$t = 40\ln\left(\frac{4}{3}\right)$$

Compare $y(t)$ at $t=0$ and $t = 40 \ln\left(\frac{4}{3}\right)$

$y(0) = 50$ (given)

$$y\left(40 \ln\left(\tfrac{4}{3}\right)\right) = -100\, e^{-\frac{t}{20}} + 150\, e^{-\frac{1}{40}t}$$

$$= -100\, e^{-\frac{1}{20} 40 \ln\left(\frac{4}{3}\right)} + 150\, e^{-\frac{1}{40} 40 \ln\left(\frac{4}{3}\right)}$$

$$= -100\, e^{-2\ln\left(\frac{4}{3}\right)} + 150\, e^{-\ln\left(\frac{4}{3}\right)}$$

$$= -100\, e^{\ln\left[\left(\frac{4}{3}\right)^{-2}\right]} + 150\, e^{\ln\left(\frac{3}{4}\right)}$$

$$= -100 \frac{3^2}{4^2} + 150 \cdot \frac{3}{4}$$

$$= -\frac{100}{4} \frac{3^2}{4} + 50 \frac{3^2}{4}$$

$$= (-25 + 50) \frac{3^2}{4}$$

$$= 25 \frac{9}{4}$$

$$= 25\left(2\tfrac{1}{4}\right)$$ is bigger than $50 = 25(2)$

$$= 50 + 25\tfrac{1}{4}$$

The max value of $y(t)$ is $\boxed{56\tfrac{1}{4} \text{ lbs.}}$

$\square$