Sec 1.5 Linear first-order equations

A <u>linear first-order differential equation</u> is of the form

$$\frac{dy}{dx} + P(x)\, y = Q(x), \quad y(x_0) = y_0$$

functions of x which are continuous on some interval I

Ex 1: $\frac{dy}{dx} = y + \frac{11}{8} e^{-\frac{x}{3}}, \quad y(0) = -1$

$$\frac{dy}{dx} \underset{P(x)=-1}{-} y = \underbrace{\frac{11}{8} e^{-\frac{x}{3}}}_{Q(x)}$$

Method for finding a solution

<u>Step 1</u> Compute integrating factor $e^{\int P(x)\,dx}$

Ex: $e^{\int -1\,dx} = e^{-x}$

<u>Step 2</u> Multiply both sides of ODE by the integrating factor

$$e^{\int P(x)\,dx} \frac{dy}{dx} + e^{\int P(x)\,dx} P(x)\, y = e^{\int P(x)\,dx} Q(x)$$

Ex: $e^{-x} \frac{dy}{dx} - e^{-x} y = e^{-x} \frac{11}{8} e^{-\frac{x}{3}}$

<u>Step 3</u> Recognize that the LHS is now the derivative of a product with respect to x, using the product rule

$$e^{\int P(x)\,dx} \frac{dy}{dx} + \underbrace{e^{\int P(x)\,dx} P(x)}\, y = e^{\int P(x)\,dx} Q(x)$$

$$e^{\int P\,dx} \cdot \frac{dy(x)}{dx} \quad \frac{d}{dx}\left(e^{\int P\,dx}\right) y(x)$$

So step 2 eq. can be rewritten as

$$\frac{d}{dx}\left[e^{\int P(x)\,dx} \cdot y(x)\right] = e^{\int P(x)\,dx} Q(x)$$

Ex: $$\frac{d}{dx}\left[e^{-x} \cdot y(x)\right] = \frac{11}{8} e^{-\frac{4}{3}x}$$

<u>Step 4</u> Integrate both sides of the equation

$$\frac{d}{dx}\left[e^{\int P(x)\,dx} \cdot y(x)\right] = e^{\int P(x)\,dx} Q(x)$$

$$e^{\int P(x)\,dx}\, y(x) = \text{integral of } e^{\int P(x)\,dx} Q(x) \text{ plus constant}$$

Ex: $$\frac{d}{dx}\left[e^{-x} \cdot y(x)\right] = \frac{11}{8} e^{-\frac{4}{3}x}$$

$$e^{-x} \cdot y(x) = \int \frac{11}{8} e^{-\frac{4}{3}x}\,dx + C$$

$$= \frac{11}{8} e^{-\frac{4}{3}x} \frac{1}{\left(-\frac{4}{3}\right)} + C$$

$$= \frac{11}{8}\left(-\frac{3}{4}\right) e^{-\frac{4}{3}x} + C$$

$$= -\frac{33}{32} e^{-\frac{4}{3}x} + C$$

Step 5 Solve for $y(x)$ and get the general solution

Ex: $y(x) = e^{x}\left[-\frac{33}{32}e^{-\frac{4}{3}x} + C\right]$ is the general solution.

Step 6 Find particular solution
(if an initial condition is given)

Ex: $y(0) = -1$ is given.

Set $x=0$, $y=-1$:

$$-1 = y(0) = e^{0}\left[-\frac{33}{32}e^{-\frac{4}{3}0} + C\right]$$

$$-1 = -\frac{33}{32} + C$$

$$\frac{1}{32} = C$$

Particular solution is $y(x) = e^{x}\left[-\frac{33}{32}e^{-\frac{4}{3}x} + \frac{1}{32}\right]$

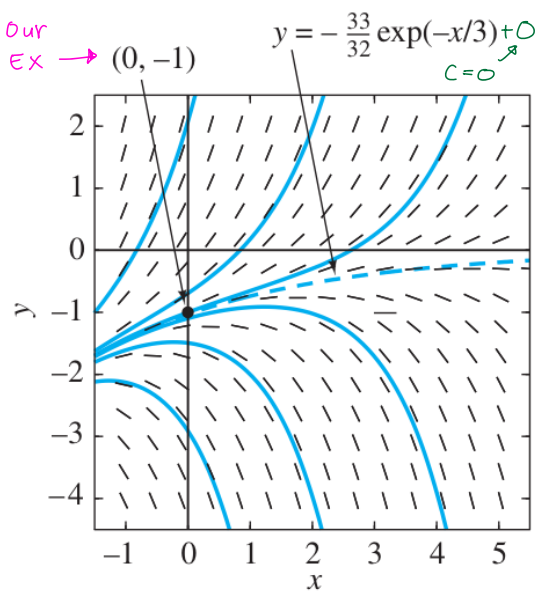

FIGURE 1.5.1. Slope field and solution curves for $y' = y + \frac{11}{8}e^{-x/3}$.

← This solution $y = e^{x}\left[-\frac{33}{32}e^{-\frac{4}{3}x}\right]$ is called critical because it "separates" solutions with very different behaviors at $+\infty$ (growing rapidly in the positive direction, or growing rapidly in the negative direction).

Theorem

(Existence & uniqueness of solutions of linear first order equations)

If the functions $P(x)$ and $Q(x)$ are continuous on the open interval I containing the point x_0, then the initial value problem IVP

$$\frac{dy}{dx} + P(x)\, y = Q(x) \quad , \qquad y(x_0) = y_0$$

has a $\underbrace{\text{unique}}_{\text{exactly one}}$ solution on I.

To find the solution, perform steps 1–6 described above.

Rem 1 The theorem guarantees that the solution is defined on the entire interval I, in contrast to the Existence & uniqueness theorem (in Sec 1.3) for ODE $\frac{dy}{dx} = f(x,y)$

Rem 2 The theorem tells us that every solution is included in the general solution (Step 5 in the method). So a linear first-order differential equation has no singular solution.

recall: a solution is singular if it's not included in the general solution

Example 2: $(x^2-1)\dfrac{dy}{dx} + 3xy = 6x$, $y(\sqrt{2})=0$

Ans Step 0 Put into $\dfrac{dy}{dx} + P(x)\,y = Q(x)$ form:

We will work with this new (different) IVP instead!

$$\frac{dy}{dx} + \underbrace{\frac{3x}{x^2-1}}_{P(x)}\, y = \underbrace{\frac{6x}{x^2-1}}_{Q(x)} \quad , \quad y(\sqrt{2})=0$$

* $P(x)$ and $Q(x)$ are continuous at every number in $\mathbb{R}$ except 1 and -1,

that is, at $(-\infty,-1)\cup(-1,1)\cup(1,\infty)$.

Since $x_0=\sqrt{2}\in(1,\infty)$, the Existence & Uniqueness solution theorem for linear 1st-order ODE guarantees that there is exactly one solution $y(x)$ in the interval $I=(1,\infty)$ such that $y(\sqrt{2})=0$.

To find this solution, use the method above, steps 1–6.

Step 1 Compute the integrating factor

$$\int P(x)\,dx = \int \frac{3x}{x^2-1}\,dx$$

$u = x^2-1$
$du = 2x$
$\frac{1}{2}du = x\,dx$

$$= \int 3\,\frac{1}{u}\,\frac{1}{2}\,du$$

$$= \frac{3}{2}\ln|u|$$

$$= \frac{3}{2}\ln|x^2-1|$$

because in $I=(1,\infty)$ $x^2-1>0$

$$= \frac{3}{2}\ln(x^2-1)$$

Integrating factor is $e^{\int P(x)dx} = e^{\frac{3}{2}\ln(x^2-1)}$

$$= e^{\ln\left[(x^2-1)^{\frac{3}{2}}\right]}$$

$$= \boxed{(x^2-1)^{\frac{3}{2}}}$$

Step 2 Multiply both sides by the integrating factor

$$(x^2-1)^{\frac{3}{2}}\frac{dy}{dx} + (x^2-1)^{\frac{3}{2}}\frac{3x}{x^2-1}\,y = (x^2-1)^{\frac{3}{2}}\frac{6x}{x^2-1}$$

$$\underbrace{(x^2-1)^{\frac{3}{2}}\frac{dy}{dx} + (x^2-1)^{\frac{1}{2}}\, 3x\; y}_{\frac{d}{dx}\left[(x^2-1)^{\frac{3}{2}}\, y(x)\right]} = (x^2-1)^{\frac{1}{2}}\; 6x$$

<u>Step 3</u> Recognize the LHS as a derivative

<u>Step 4</u> Integrate both sides

$$(x^2-1)^{\frac{3}{2}}\, y(x) = \int (x^2-1)^{\frac{1}{2}}\; 6x \; dx + C$$

$u = x^2-1$

$du = 2x\; dx$

$\frac{1}{2} du = x\, dx$

$$= \int u^{\frac{1}{2}}\, 6\, \frac{1}{2}\, du + C$$

$$= \frac{3\; u^{\frac{3}{2}}}{\left(\frac{3}{2}\right)} + C$$

$$= 2\; u^{\frac{3}{2}} + C$$

$$= 2\,(x^2-1)^{\frac{3}{2}} + C$$

<u>Step 5</u> Solve for $y(x)$:

$$y(x) = 2\,\frac{(x^2-1)^{\frac{3}{2}}}{(x^2-1)^{\frac{3}{2}}} + \frac{C}{(x^2-1)^{\frac{3}{2}}}$$

$$y(x) = 2 + \frac{C}{(x^2-1)^{\frac{3}{2}}} \quad \text{for } x \in (1, \infty), \quad C \in \mathbb{R}$$

<u>Step 6</u> Impose the initial condition $y(\sqrt{2}) = 0$:

Set $x = \sqrt{2}$, $y = 0$

$$0 = y(\sqrt{2}) = 2 + \frac{C}{(2-1)^{\frac{3}{2}}}$$

$$0 = 2 + C$$

$$-2 = C$$

The unique solution of the IVP $\frac{dy}{dx} + \frac{3x}{x^2-1}\, y = \frac{6x}{x^2-1}$, $y(\sqrt{2}) = 0$ is

$$y(x) = 2 - \frac{2}{(x^2-1)^{\frac{3}{2}}} \quad \text{for } x \text{ in } (1, \infty)$$