Sec 1.4 (a) Separable equations (b) Applications

(b) Applications : Useful equations that can be solved by separation of variables

Natural growth / decay ODE

$$\frac{dx}{dt} = kx \,, \qquad x(t) > 0$$

where $k \in \mathbb{R}$ is a fixed constant

Growth rate of a quantity $x(t)$ is proportional to $x(t)$.

Here our variables are x, t

E.g. $\frac{dx}{dt} = 0.1\, x$ is a natural growth ODE.

Can solve using separation of variables :

Step 1 Separation by variables

$$\underbrace{\frac{1}{x}\, dx}_{\text{only } x} = \underbrace{k\, dt}_{\text{only } t}$$

Step 2 Integrate both sides to find implicit general solution

$$\int \frac{1}{x}\, dx = \int k\, dt$$

$$\ln|x| = kt + C$$

$$\ln(x) = kt + C$$ since $x(t) > 0$ by assumption

Step 3 Find an explicit solution.

$$e^{\ln(x)} = e^{kt + C}$$

$$x(t) = e^{kt} \underbrace{e^{C}}_{D > 0}$$

General solution

$$\boxed{x(t) = D e^{kt} \,, \quad D > 0}$$

Rem Because of this we call $\frac{dx}{dt} = kx$ the exponential equation/ natural growth equation

Step 4 If we impose initial condition $x(0) = x_0$, then we can find D:

Set $t=0$, $x = x_0$:

$$x(0) = D e^{k0} = x_0$$

$$D\,1 = x_0$$

So the solution of the IVP $\frac{dx}{dt} = kx$, $x(0) = x_0$

is $x(t) = x_0 e^{kt}$.

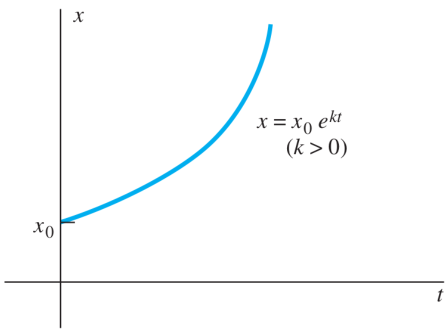

FIGURE 1.4.7. Natural growth.

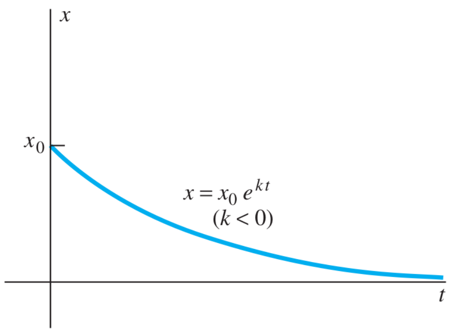

FIGURE 1.4.8. Natural decay.

Rem If the value for k is not given, then we can compute k from knowing the value of $x(t)$ for some t.

$\boxed{\text{Compound interest}}$

$A(t)$: dollars in a high-yield savings account at year t with annual interest rate $10\% = \frac{1}{10}$.

Assume the interest is compounded continuously

during a short time interval Δt, the amount of interest added to the account is approximately $\frac{1}{10} A(t)(\Delta t)$.

Then $A(t)$ satisfies the equation $\frac{dA}{dt} = 0.1\, A$

$\boxed{\text{Drug elimination}}$ Extra example

$A(t)$: amount of excess of certain drug in the blood stream (over the natural level)

In many cases, $A(t)$ satisfies the ODE

$$\frac{dA}{dt} = -\lambda A$$

excess amount declines at a rate proportional to current excess amount

$\lambda > 0$ is called the <u>elimination constant</u> of the drug

$\boxed{\text{Population growth :}}$

$P(t)$: number of individuals in a population.

β : birth rate, we suppose it to be constant.

δ : death rate, we suppose it to be constant.

Then
$$\frac{dP}{dt} = \underset{\text{births}}{\beta P} - \underset{\text{deaths}}{\delta P}$$

proportional to P

The rate of change of P at a time t is the number of births minus the number of deaths

$$\frac{dP}{dt} = \underbrace{(\beta - \delta)}_{k} P$$

It is the exponential equation with $k = \beta - \delta$.

Example: The world's population reached 6 billion persons in mid-1999 and was increasing at that moment at a rate of 212 000 persons each day. We assume that natural population growth at this rate continues.

This means that if $P(t)$ is the population (in billions) at time t (in years), then $P(t)$ satisfies the natural growth equation for some $k>0$:

$$P' = \frac{dP}{dt} = kP.$$

First, we can find k: we know that $P(0) = 6$ and

$$P'(0) = \underbrace{0.000212}_{\text{rate of change per day}} \cdot \underbrace{365.25}_{\text{days in a year}} = \underbrace{0.07743}_{\text{rate of change per year (our unit of time)}}$$

Then from $P'(0) = kP(0)$ we deduce that

$$k = \frac{P'(0)}{P(0)} = \frac{0.07743}{6} = 0.0129$$

This means that the world population was growing at the rate of 1.29% anually in 1999.

Then, $P(t) = x_0 e^{kt} = 6e^{0.0129t}$.

We can estimate the population in 2050 (then $t=51$ since 1999 is $t=0$):

$$P(51) = 6\, e^{0.0129 \cdot 51} \approx 11.58 \quad (\text{billion}).$$

We can also estimate when the population will be 60 billion:

$$P(t) = 6\, e^{0.0129t} = 60$$

$$e^{0.0129t} = 10$$

$$0.0129\, t = \ln(10)$$

$$t = \frac{\ln(10)}{0.0129} \approx 178 \quad (\text{In the year } 2177).$$

See also example 4 in the textbook.

$\boxed{\text{Cooling and heating}}$

Newton's law of cooling: rate of change of temperature $T(t)$ of an object immersed in a medium of constant temperature A is proportional to the temperature difference $A - T(t)$,

i.e. $$\frac{dT}{dt} = k\,(A - T)$$

k is a positive constant

Example

A meal, initially at 50°F, is placed in an oven (pre-heated to 375°F). After 75 minutes the temperature of the meal is 125°F. When will the meal be 150°F?

Ans

$T(t)$: temperature of meal after t minutes

So $T(0) = 50$, $T(75) = 125$.

A : temperature of the medium (oven), $A = 375$.

Note : $\underbrace{T(t)}_{\text{meal temp}} < \underbrace{A = 375}_{\text{oven temp}}$

$$\frac{dT}{dt} = k(\underbrace{375}_{A} - T)$$ due to Newton's law of cooling

Step 1 & 2 & 3 Separation by variables & integrate both sides & find explicit general solution to $\frac{dT}{dt} = k(375-T)$

$$\int \frac{1}{375-T}\, dT = \int k\, dt$$

$$-\ln|375-T| = kt + C$$

$$-\ln(375-T) = kt + C$$ because $T(t) < 375$

$$e^{\ln(375-T)} = e^{-kt-C}$$

$$375 - T = e^{-kt} \underbrace{e^{-C}}_{\text{Let } B = e^{-C}}$$ positive for all $C \in \mathbb{R}$

$$375 - Be^{-kt} = T(t)$$

Step 4 Impose initial condition $T(0) = 50$ to find B.

Set $t=0$, $T=50$:

$$T(0) = 375 - Be^{-k.0} = 50$$

$$325 = B.1$$

Additional Step Find k.

We were given $T(75) = 125$.

Set $t = 75$, $T = 125$:

$$T(75) = 375 - 325\, e^{-k75} = 125$$

$$\cancel{250} = \cancel{325}\, e^{-k75}$$ (You don't need to simplify)

$$\frac{10}{13} = e^{-k75}$$

$$\ln \frac{10}{13} = -k75$$

$$-\frac{1}{75} \ln\left(\frac{10}{13}\right) = k$$

So $T(t) = 375 - 325\, e^{\frac{1}{75}\ln\left(\frac{10}{13}\right) t}$

When will the meal be 150° F?

Set $T(t) = 150$

$$375 - 325\, e^{\frac{1}{75}\ln\left(\frac{10}{13}\right) t} = 150$$

$$225 = 325\, e^{\frac{1}{75}\ln\left(\frac{10}{13}\right) t}$$

$$\frac{9}{13} = e^{\frac{1}{75}\ln\left(\frac{10}{13}\right) t}$$

$$\ln \frac{9}{13} = \frac{1}{75} \ln\left(\frac{10}{13}\right) t$$

$$\underbrace{75\, \frac{\ln\left(\frac{9}{13}\right)}{\ln\left(\frac{10}{13}\right)}}_{\text{about 105 mins}} = t$$

The meal will be 150°F after 105 mins. $\square$