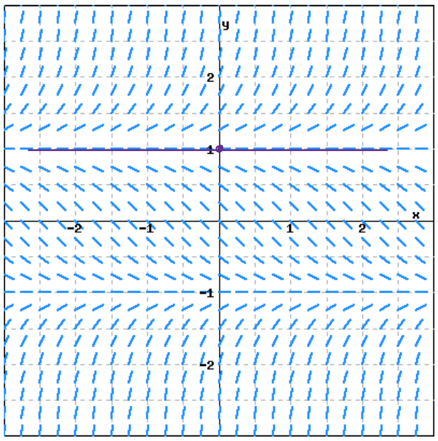
Review Sec 1.3 WW02 Prob 6

1) ~~(b)~~ For the solution that satisfies $y(0) = 1$, sketch the solution curve and estimate the following: $y(1) \approx$ 1 and $y(-1) \approx$ 1



2) What are the equilibrium solutions (if any)?

Ans $y(x) = 1$ and $y(x) = -1$

Rem: line segments are all horizontal (slope 0) when $y=1$ and $y=-1$.

↑ Sketch of the slope field of an ODE

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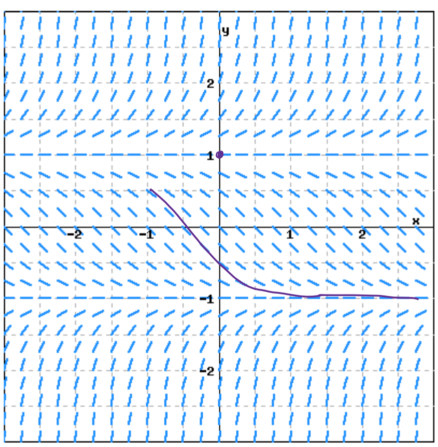
3) Sketch the solution such that $y(-1) = 0.5$

4) For this solution, what is $\lim_{x \to \infty} y(x)$? Ans -1

5) For this solution, what is $\lim_{x \to -\infty} y(x)$? Ans 1



Sec 1.4   (a) Separable equations (b) Applications

(a) New Method: Separation of variables

Running Example

Consider the ODE $\dfrac{dy}{dx} = \underbrace{-6xy}_{\text{function of } x \text{ and } y} = \overbrace{(-6x)}^{\text{function only in } x}\ \underbrace{y}_{\text{function only in } y}$

Step 1   Informally "separate the variables"

$$\underbrace{\frac{1}{y}\,dy}_{\text{all the } y\text{'s on one side}} = \underbrace{-6x\,dx}_{\text{all the } x\text{'s on the other side}}$$

Step 2   Integrate both sides

$$\int \frac{1}{y}\,dy = \int -6x\,dx$$

$$\ln|y| + C_1 = \frac{-6x^2}{2} + C_2$$

$$\ln|y| = -3x^2 + \underbrace{(C_2 - C_1)}$$

we only need to write one constant $C = C_2 - C_1$

$$\boxed{\ln|y| = -3x^2 + C}$$

This is the implicit general solution of the ODE

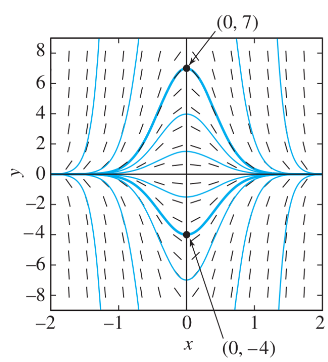


FIGURE 1.4.1. Slope field and solution curves for $y' = -6xy$.

Solution curves for a few values of C

Step 3 If possible, find an explicit general solution $y(x)$:

Ex (con't) Take the exponential of both sides

$$e^{\ln|y|} = e^{-3x^2 + C}$$

$$|y| = e^{-3x^2} e^C \text{ — always positive for } C \in \mathbb{R}$$

"in"

meaning: "C is a real number"

$$y = e^C e^{-3x^2} \quad \text{or} \quad y = -e^C e^{-3x^2}$$

I.e. $y(x) = \pm e^C e^{-3x^2}$

I can write $A := \pm e^C$, either positive or negative (not 0)

Ie $\boxed{y(x) = A e^{-3x^2} \quad \text{where } A \neq 0}$ — explicit general solution

Step 4 Impose an initial condition (if it's given), e.g. $y(0) = 7$

Ex (con't)

$$y(0) = A e^{-3 \cdot 0^2} = 7$$
$$A \cdot 1 = 7$$
$$A = 7$$

The particular solution for the IVP $\frac{dy}{dx} = -6xy$, $y(0) = 7$ is $\boxed{y(x) = 7e^{-3x^2}}$

Exercise: Can we use the Existence & Uniqueness Thm (Sec 1.3) to guarantee that this solution is in fact the only solution to this IVP? Yes or No?

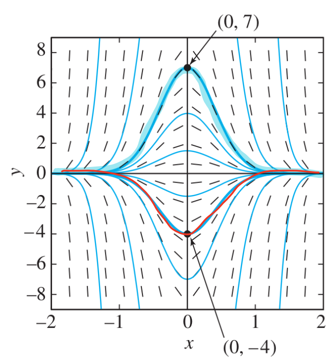


FIGURE 1.4.1. Slope field and solution curves for $y' = -6xy$.

Ex: If the initial condition is $y(0) = -4$?

Ans we get the particular solution $\boxed{y(x) = -4e^{-3x^2}}$

Def

In general, a 1st-order ODE $\frac{dy}{dx} = g(x)\,k(y) = \frac{g(x)}{h(y)}$ is called separable.

Ex Separable equations

1. $\frac{dy}{dx} = y^2$

2. $\frac{dy}{dx} = \frac{y}{x^2}$

3. $\frac{dy}{dx} = \frac{1}{x^2}$

4. $\frac{dy}{dx} = e^{3x+2y} = e^{3x}\,e^{2y}$

5. $\frac{dy}{dx} = \frac{4-2x}{3y^2-5}$

6. $\frac{dy}{dx} = 10+4x+15y+6xy$
$= 2(5+2x)+3y(5+2x)$
$= (2+3y)(5+2x)$

9. $\frac{dy}{dx} + y\cos(x) = 5\cos(x) \iff \frac{dy}{dx} = \cos x(-y+5)$

Not separable

7. $\frac{dy}{dx} = y^2 + x$

8. $\frac{dy}{dx} = \sin(xy)$

In general, for separable ODE $\frac{dy}{dx} = \frac{g(x)}{h(y)}$

Step 1 Separate the variables

$$h(y)\,dy = g(x)\,dx$$

Step 2 Integrate both sides

$$\underbrace{\int h(y)\,dy}_{} = \underbrace{\int g(x)\,dx}_{} + C$$

let $H(y)$ be an antiderivative of $h(x)$

let $G(x)$ be an antiderivative of $g(x)$

$C \in \mathbb{R}$

Get the implicit general solution of $\frac{dy}{dx} = \frac{g(x)}{h(x)}$: $\boxed{H(y) = G(x) + C}$ $C \in \mathbb{R}$

C is any number

Def An equation is called an implicit solution of an ODE if it is satisfied (on some interval) by some solution of the ODE.

Downside of separation of variable methods

Issue 1 may miss solutions

Ex (cont)

You can verify $y(x) \equiv 0$ is a solution to the ODE $\frac{dy}{dx} = -6xy$

(by substitution method from Sec 1.1

or by looking at the slope field),

but $y(x) \equiv 0$ is not included in the general solution

$y(x) = A e^{-3x^2}$, $A \neq 0$.

Why did we miss this solution?

We turned $\frac{dy}{dx} = -6xy$

into $\frac{1}{y}\frac{dy}{dx} = -6x$

Now y cannot be 0.

So $y(x) \equiv 0$ is not a solution to $\frac{1}{y}\frac{dy}{dx} = -6x$

Def A singular solution of an ODE is a particular solution which is not contained in a general solution of the ODE.

Issue 2   Implicit solutions may be "too large"

Ex IVP $\underbrace{\frac{dy}{dx} = y^2}_{\text{ODE}}$, $\underbrace{y(2) = -1}$

initial value

Step 1   Separation of variables:

$$\frac{1}{y^2}\frac{dy}{dx} = 1$$

$$\frac{1}{y^2}\,dy = dx$$

Step 2   Integrate both sides:

$$\int \frac{1}{y^2}\,dy = \int dx$$

$$-\frac{1}{y} = x + C \qquad C \in \mathbb{R}$$

implicit general solution

Step 3   Find an explicit general solution:

$$y(x) = -\frac{1}{x+C}, \quad \underbrace{C \in \mathbb{R}}_{\text{for } C \text{ any real number}}$$

Student's Question: Do we say $C \neq x$?

Answer No, we say for any real number C

Step 4   Impose the initial condition:

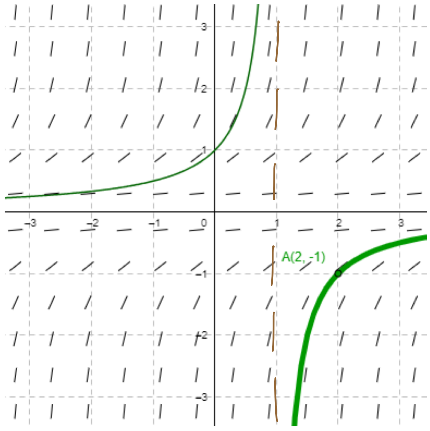
set $x = 2$, $y = -1$

$$y(2) = -\frac{1}{2+C} = -1$$

$$1 = 2 + C$$

$$C = -1$$

Then $y(x) = -\frac{1}{x-1} = \frac{1}{1-x}$ not continuous at $x = 1$



vertical asymptote at $x=1$

- A solution of a differential equation must be differentiable, and hence continuous.

- So $y(x) = \frac{1}{1-x}$ is a solution to the IVP $\frac{dy}{dx} = y^2$, $y(2) = -1$ but only in an interval where it is differentiable.

- $y(x) = \frac{1}{1-x}$ is differentiable in $(-\infty, 1)$ and $(1, +\infty)$.

- $y(x) = \frac{1}{1-x}$ is a solution of the IVP on $(1, +\infty)$, but not on $(-\infty, 1)$.

- The method provided a "too large solution" in the sense that we had to restrict the function to an interval where it is differentiable.

Rem
- Verify that $y(x) = 0$ is a solution to $\frac{dy}{dx} = y^2$
- It's not a solution of the form $y(x) = -\frac{1}{x+C}$ for any $C \in \mathbb{R}$, So $y(x) = 0$ is an example of a singular solution

BONUS EXAMPLE

Ex (Webwork WW03) Prob 6

Find solution to IVP $\frac{dy}{dx} = (y-1)(y+1)$, $y(4)=0$

Step 1 Separation of variables

$$\frac{1}{(y-1)(y+1)}\,dy = dx$$

Step 2 Integrate both sides & find implicit general solution

$$\int \frac{1}{(y-1)(y+1)}\,dy = \int dx$$

Partial fraction:

$$\frac{1}{(y-1)(y+1)} = \frac{A}{y-1} + \frac{B}{y+1}$$

$$1 = A(y+1) + B(y-1)$$

$$0y + 1 = (A+B)y + (A-B)$$

and $\left.\begin{array}{l} 0 = A+B \\ 1 = A-B \end{array}\right\} \xrightarrow[\text{(do algebra)}]{} A = \frac{1}{2},\ B = -\frac{1}{2}$

So $\frac{1}{(y-1)(y+1)} = \frac{1}{2}\frac{1}{(y-1)} - \frac{1}{2}\frac{1}{(y+1)}$

$$\int \frac{1}{2}\frac{1}{(y-1)} - \frac{1}{2}\frac{1}{(y+1)}\,dy = \int dx$$

$$\frac{1}{2}\ln|y-1| - \frac{1}{2}\ln|y+1| = x + C$$

$$\boxed{\ln|y-1| - \ln|y+1| = 2x + D, \quad D \in \mathbb{R}}$$

D is a real number

Implicit general solution

(con't) →

Step 3 Find explicit general solution $y(x)$, if possible:

$$e^{\ln|y-1| - \ln|y+1|} = e^{2x+D}$$

$$\frac{e^{\ln|y-1|}}{e^{\ln|y+1|}} = e^{2x+D}$$

$$\frac{|y-1|}{|y+1|} = e^{2x+D}$$

$$\left|\frac{y-1}{y+1}\right| = e^{2x}e^{D}$$

$$\frac{y-1}{y+1} = \underbrace{\pm e^{D}}\, e^{2x}$$

Let $A = \pm e^{D}$, so $\boxed{A \in \mathbb{R} \setminus \{0\}}$

(This is the same as "A can be any real number except 0")

$$y-1 = Ae^{2x}(y+1)$$

$$y-1 = Ae^{2x}y + Ae^{2x}$$

$$y - Ae^{2x}y = 1 + Ae^{2x}$$

$$y(1-Ae^{2x}) = 1+Ae^{2x}$$

$$\boxed{y(x) = \frac{1+Ae^{2x}}{1-Ae^{2x}}, \quad A \in \mathbb{R}\setminus\{0\}}$$

Explicit general solution

Step 4 Impose initial value: Set $x=4$, $y=0$

$$y(4) = \frac{1+Ae^{8}}{1-Ae^{8}} = 0$$

$$\left.\begin{aligned} 1+Ae^{8} &= 0 \\ \text{and} \quad 1-Ae^{8} &\neq 0 \end{aligned}\right\} \dashrightarrow Ae^{8} = -1 \dashrightarrow A = -\frac{1}{e^{8}}$$

$$y(x) = \frac{1+(-e^{-8})e^{2x}}{1-(-e^{-8})e^{2x}}$$

multiply by $1 = \frac{e^{8}}{e^{8}}$ to make the function look nice

Solution to the IVP

$$\boxed{y(x) = \frac{e^{8} - e^{2x}}{e^{8} + e^{2x}}}$$